Journal of Mathematics and Music, 20XX
Vol. 9999, No. 9999

Diatonic Minimum Spread Tuning

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A temperament holding a lowest diatonic C-major interval beating rate spread on fifths, and major and minor thirds was calculated. It appears that beating rate equality is a primordial quality factor for auditory tuning. Fortunately, no significant difference can be encountered between tuning pitches based on comma division, ratios or cents, and those obtained based on equal interval beating rates. It appears Vallotti and a new developed well tempered meantone are by far the temperaments with lowest impurity spread. In conclusion, it is probably right to assume that the auditory tuning of keyboards was the basis for the development of historical temperaments, although the results were often described mathematically on the basis of proportional relationships.

Keywords: keyboard; auditory; tuning; diatonic; interval; beating rate; spread; circular; temperament; Vallotti; meantone;

Observation: The present text is the draft that was submitted to the "Journal of Mathematics and Music", slightly repleted here in the introduction and par. 10, "Closing thoughts" (|*see italic text*|).

https://doi.org/10.1080/17459737.2024.2398425

This paper has to be considered as a minor but probably interesting and specific supplement to major publications such as those of | *Barbour (1951)* |; Kelletat (1960, 1981, 1982, 1994); Rasch (1984, 1983); Jedrzejewski (2002); Sethares (2005), . . . up to more recent ones like Di Veroli (2009).

Auditory tuning involves initial setting of musical intervals within one octave. Those intervals inherently deviate from perfect consonance, for there are more musically significant intervals than defined notes. This led to numerous historical musical temperaments.

Octaves are supposed to have a perfect 2/1 ratio, and each note in itself should be a perfect prime, ratio 1/1, but even well-tuned octaves and primes may have slightly beating sounds, due to inhomogeneity of vibrating bodies and differences in inharmonicity.

Quite some historical keyboard tuning instructions publish information on auditory interval quality, backed by little or no quantitative information, but many more are formulated based on interval ratios, comma divisions or cents, whereby the auditory part of the process was not always clear. There is some kind of "chicken or egg" tuning problem: "what comes first: the auditory experience, or the measurable instructions?"
Many historic and measurable instructions are based on the monochord. von Helmholtz (1863) still relied on physical resonators for sound analysis. It is clear, today, that the precision of those ancient equipments was rather low. The inharmonicity of a string, for example, can be approximately calculated based on a differential equation of the

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fourth order, and rapidly exceeds a factor 1.001, what corresponds with 1.7 cents, and it increases further with the order of the partials. Following formula enables to predict the inharmonicity of strings Fletcher H. (1962, vol. 34, p. 749–761).

$$f(n) = f(0).n.\sqrt{1+B.n^2} \quad \text{with} \quad B = \frac{\pi^3.E.d^4}{64.l^2.T} \quad \text{E=Young's modulus. T=tension}$$

A Fourier analysis on a sample of more than 50 cycles of the C♯3 string of a Bruckner stand up piano reveals an inharmonicity of 4 cents already on the second partial and more on higher ones; the middle range string G♯4 has an inharmonicity of 11 cents already on its second partial. A Fourier analysis on a 0.5 seconds sample of an A3 string of a Hanlet stand up piano (Belgium), up to its seventh partial, –the first six import for setting the initial F3–F4 partition–, is displayed in the table below.

2	3	4	5	6	7
1.00000	1.00076	1.00171	1.00227	1.00576	1.00630

Inharmonicities Hanlet A3 string.

The human ear characteristics also influence keyboard tuning, for its sensitivity strongly depends on frequency Fletcher H. (1933, vol, 5, p. 82–108), and therefore the partials outside the 300 to 3000 Hz band have lower influence,... very low or high frequency partials may at best be weighted to limit their influence, if involved in measurements or calculations. Combination of all the above factors results in the elongation of the octave Railsback (1938, vol. 9, p. 274),

Because of all the above, and up to a recent past, most keyboard tuning was almost always auditory only, *for there was no other smoothly workable way.*

It is only quite recent that better electronic pitch measuring equipment has become available, enabling very precise measurements, and therefore very precise setting of pitches too, unfortunately favoring the 12–TET (...because of ignorance). But even so, and at the very least for achieving good high tones octave consonance, those instruments should also measure partials for correct interval setting. Especially for very low tones, on the other hand, there is an additional problem to find out how partials should be taken into account, because of necessary counting with ear sensitivity, high harmonic content paired with heavy inharmonicity,...the latter conditions being typical and essential for those tones.

1. Possible determinating factors in auditory tuning

Superficial analysis of historic temperaments is sufficient to find out that most proposed temperaments aim for better quality of the diatonic C–major tonality, and that in general, this is achieved by diminution of diatonic fifths, in order to improve the quality of diatonic thirds, but in such a way that sufficient quality is maintained for all other tonalities. The tuning can range from perfect fifths with Pythagorean thirds to diminuated fifths with a just major third. The question is : how does an auditory tuner intuitively assess the quality ?

2. Hypothesis based on historical development

The meantone introduced the better major thirds. The chronology of some meantone alternatives might be revealing, although it must be admitted chronology differs in differing regions, and the here used chronology can therefore be subject of discussion. The meantone started with the 1/4 comma version Aaron (1523), later on we have the 1/5 comma version Sauveur (1701), Rossi (1666), and thereafter the 1/6 comma version Romieu (1758). Our hypothesis is that those were not created based on comma division using a monochord, but possibly by auditory tuning as follows.

2.1. *The 1/4 meantone version, Di Veroli (2018), Fogliano (1529)*

2.1.1. The comma division with A as diapason might have been done by following steps

(1) Tune a pure major third F–A
(2) Tune a perfect fifth F–C and a perfect fifth D–A
(3) Tune G to have equally diminuated fifths C–G and G–A; both thus become flat by 1/2 comma
(4) Retune C to have to equally diminuated fifths F–C and C–G ; therefore both will become flat by halve a 1/2 comma, it is to say, effectively 1/4 comma
(5) Do the same by retuning D, to have two equally diminuated fifths G–D and D–A

2.1.2. The comma division with C as diapason might have been done by following steps

(1) Tune a pure major third C–E
(2) Tune a perfect fifth C–G and a perfect fifth A–E
(3) Tune D to have equally diminuated fifths G–D and D–A; both become flat by 1/2 comma
(4) Retune G to have to equally diminuated fifths C–G and G–D ; therefore both thus become flat by halve a 1/2 comma, it is to say, effectively 1/4 comma
(5) Do the same by retuning A, to have two equally diminuated fifths D–A and A–E

Both procedures result in ***four equally beating flat fifths, and a pure major third***. In succeeding routine practice, as the required fifths beating rate became known, tuning will rather be achieved by setting the four fifths, followed by control of the major third. Classic and "beating rate equality" pitches are compared in the table below.

Note	F3	C4	G3	D4	A3	E4		Beating rate
Classic	176.00	263.18	196.77	294.25	220.00	328.98		–
Beat	176.00	263.12	196.78	294.07	220.00	328.89		−2.21

Further meantone tuning is achieved, setting seven more pure major thirds.

2.2. *The 1/5 meantone version*

The 1/4 meantone fifth has a strong wolf fifth. It is possible to reduce its dissonance by sharpening the major thirds a little. This is possible following the procedure below, starting with fifths that are slightly better than the 1/4 comma fifths.
Based on the above experience, and by trial and error, create a major third based on four flat fifths, all those five intervals holding an ***equal absolute beating rate*** (note: 1+1+1+1+1=5).

Note	F3	C4	G3	D4	A3	E4		Beating rate
Classic	175.56	262.69	196.53	294.06	220.00	329.18		–
Beat	175.61	262.83	196.64	293.98	220.00	329.03		−1.95

2.3. *The 1/6 meantone version*

The 1/5 meantone fifth still holds a strong wolf fifth. Further reduction of its dissonance can by sharpening the major thirds a little more. It is possible following the procedure below starting with fifths that are slightly better than the 1/5 comma fifths.
Based on the above experience, and by trial and error, create a major third based on four flat fifths, those four fifths having ***equal absolute beating rates, equal to half the beating rate of the associated major third*** (note: 1+1+1+1+2=6).
Classic and "beating rate equality" pitches are compared in the table below.

Note	F3	C4	G3	D4	A3	E4		Beating rate
Classic	175.27	262.37	196.37	293.94	220.00	329.32		–
Beat	175.30	262.61	196.52	293.91	220.00	329.13		–1.74

Observation
All three meantone temperaments are easy to tune auditorily. 1/4 Meantone: four equal beating rate fifths and eight just major thirds; 1/5 meantone: four fifths and eight major thirds, all having an equal beating rate; 1/6 meantone: four equal beating rate fifths and eight major thirds with double beating rate.
A monochord verification of the resulting pitches, can with no problem at all lead to the "classic" defined comma divisions, not being aware of measuring errors.
Based on the above reasoning, and on unnoticed measuring errors, almost the full musicologic proportional comma division theory can be questioned, if not rejected, at least what concerns the meantone temperaments.

3. Minimisation of the global impurity

The natural harmonic system is built on perfect fifths and pure major and minor thirds. This is ideal, if it were not that it holds two important impure intervals: a fifth and a minor third on D, both being a full syntonic comma out of tune; 21.5 cents. This scale needs to be tempered, so it appears logic to strive for a minimum over all impurity: the sum of the impurities of the six fifths, three major thirds and four minor thirds should be as small as possible. For auditory tuning, this signifies that the sum of the beating rates of all those intervals should be minimised.
The corresponding temperament can rather easily be calculated analytically, but the obtained diatonic fifths beating rates are quite high, between ≈ -2 and ≈ -5 bps, the diatonic major thirds are diminuated, and fifths on B and altered notes beat at $\approx +1$ bps, leading to major thirds lager than pythagorean. It is clear as well, that an auditory tuner, not using any instrument or interval beating rate table, has no means to determinate whether he has achieved a minimum.
A similar calculation is possible, excluding minor thirds. The result is better, but the diatonic fifths have diverging beating rates between ≈ -0.9 and ≈ -3.2 bps, and therefore a calculated interval beating rate table is required in support of auditory tuning. The fifths on B and altered notes beat at $\approx +0.2$ bps.
No historic or antecedent publication or tuning table in support of the above procedures was found.

4. Minimisation of impurity spread

In the process of auditory tuning, an approximate equality of beating rates can rather easily be estimated, using no measuring means at all. Many tuning instructions propose an equal division of a comma, an easy mathematical requirement indeed. But it is not easy to divide a comma on a keyboard. It is possible with a monochord, but at the expense of much time and labor, including auditory comparisons for transposition of the obtained pitches between monochord and keyboard string. Moreover, the monochord lacks of sufficient precision, because of string inharmonicity. A possible alternative consists in dividing the comma by setting equal beating rates on every comma division. It is rather easy to recalculate historical temperaments based on the above assumption. Recalculations were done for a number of temperaments, among those famous ones like Werckmeister III, Neidhardt (4 ones), Kirnberger III, Vallotti, the Equal Temperament, and also meantone ones, like the quarter meantone, Silberman, etc ... It has been found that the pitches obtained by “classic” calculation, and the ones obtained for equal beating rate calculations, are very comparable: almost all calculated results display maximum differences of not more than 1 cent associated with a much lower mean difference, whereby only very few show slightly higher figures, exceptionally with a maximum of up to 2 cents. The obtained results therefore show evidence that the difference between auditory tuning and tuning based on ratios, commas or cent measurements are neglectable. This is not surprising: the difference between acceptable beating rates and the cents deviations concern small differences of small deviations; it is a kind of difference of the second order. The errors of the used measuring means –i.e. the monochord– are of the same order of magnitude, obviating the possibility to detect said possible differences. As equality has been important for equal comma division, and for it appears there is no significant difference between beating rate and cents determination of temperaments, it might be logic to strive for a temperament holding a minimal beating rate spread of impurities.

5. Calculation of minimum impurity spread

Professional auditory tuning is usually initiated within the F3–F4 partition, Calvet (2020), based on beating rates of fifths, and some thirds, striving for a good diatonic C–major quality. Reasons to select the F3–F4 partition might be historical, but also “technical”, for those strings are the lowest unwound strings, and therefore those of best possible quality. An acceptable quality of all but the C–major diatonic tonalities depends on control of the remaining fifths. The important interval beating rates for control of the diatonic C–major intervals within the F3–F4 partition, can be calculated based on the following formulas, although it should be clear that those formulas are a very primitive measure only, concerning low initial impurities, for those give no measure on the consonance window Plomp and Levelt (1965), and do not include string or sound inharmonicity factors.

Table 1. Fifths beating rates within F3–F4

$q_F = 2C4 - 3F3$	$q_C = 4G3 - 3C4$	$q_G = 2D4 - 3G3$	$q_D = 4A3 - 3D4$
$q_A = 2E4 - 3A3$	$q_E = 4B3 - 3E4$	$q_B = 4F\sharp3 - 3B3$	$q_{F\sharp} = 2C\sharp4 - 3F\sharp3$
$q_{C\sharp} = 4G\sharp - 3C\sharp4$	$q_{C\sharp} = 2E\flat4 - 3G\sharp3$	$q_{E\flat} = 4B\flat3 - 3E\flat4$	$q_{B\flat} = 4F3 - 3B\flat3$

Determination of an optimal diatonic C–major is possible without involvment of altered notes, and the evolution of deviations is so, that fifths and major thirds can grow

Table 2. Major thirds beating rates within F3–F4

$p_F = 4A3 - 5F3$	$p_C = 4E4 - 5C4$	$p_G = 4B3 - 5G3$

Table 3. Fifths beating rates within F3–F4

$r_D = 10F3 - 6D4$	$r_A = 5C4 - 6A3$	$r_E = 10G3 - 6E4$	$r_B = 5D4 - 6B3$

towards equal ***absolute*** values, while minor thirds still hold larger absolute value deviations. The spread can therefore at best be calculated with differing mean values for the group of fifths and major thirds, the group of minor thirds, and the group of fifths on B3 and on altered notes. It is necessary though, absolute impurity values should be used, for fifths have negative impurities, and major thirds have positive ones. Hence, the deviations from the absolute values of the mean, are obtained by the expressions below.

C–major diatonic fifths. $\varphi_{Note} = -q_{Note} - \frac{-q_{F3}-q_{C4}-q_{G3}-q_{D4}-q_{A3}-q_{E4}+p_{F3}+p_{C4}+p_{G3}}{9}$

C–major major thirds. $\theta_{Note} = p_{Note} - \frac{-q_{F3}-q_{C4}-q_{G3}-q_{D4}-q_{A3}-q_{E4}+p_{F3}+p_{C4}+p_{G3}}{9}$

C–major minor thirds. $\vartheta_{Note} = r_{Note} - \frac{r_{D4}+r_{A3}+r_{E4}+q_{B3}}{4}$

C–major fifths on B3 and altered notes. $q = q_B = q_{F\sharp} = q_{C\sharp} = q_{G\sharp} = q_{E\flat} = q_{B\flat}$

The sum of the diatonic C–major impurities equals.

$$\Sigma(squares) = \varphi^2_{F3}+\varphi^2_{C4}+\varphi^2_{G3}+\varphi^2_{D4}+\varphi^2_{A3}+\varphi^2_{E4}+\theta^2_{F3}+\theta^2_{C4}+\theta^2_{G3}+\vartheta^2_{D4}+\vartheta^2_{A3}+\vartheta^2_{E4}+\vartheta^2_{B3}$$

The above $\Sigma(squares)$, expressed in function of the notes within the F3–F4 partition becomes.

$$\begin{aligned}1296.\Sigma(squares) &= 140688.F_3^2+71244.C_4^2+156816.G_3^2+95436.D_4^2+86832.A_3^2+68976.E_4^2\\ &+76464.B3^2 - 50256.F_3.C_4 - 68256.F_3.G_3 - 148464.F_3.D_4 - 11232.F_3.A_3 + 41760.F_3.E_4\\ &+38880.F_3.B_3 - 70416.C_4.G_3 + 4392.C_4.D_4 - 54864.C_4.A_3 - 26640.C_4.E_4 + 19440.C_4, B_3\\ &-7344.G_3.D_4 + 44064.G_3.A_3 - 108000.G_3.E_4 - 12960.G_3.B_3 - 35856.D_4.A_3 - 5328.D_4.E_4\\ &-81648.D_4.B_3 - 43200.A_3.E_4 - 23328.A_3.B_3 - 54432.E_4.B_3\end{aligned}$$

The partial derivatives. set to zero, become after simplification of coefficients.

$$\begin{aligned}
\frac{\partial\Sigma}{\partial F_3} &= 0 \iff 1954.F_3 - 349.C_4 - 474.G_3 - 1031.D_4 - 78.A_3 + 290.E_4 + 270.B_3 = 0\\
\frac{\partial\Sigma}{\partial C_4} &= 0 \iff -698.F_3 + 1979.C_4 - 978.G_3 + 61.D_4 - 762.A_3 - 370.E_4 + 270.B_3{=}0\\
\frac{\partial\Sigma}{\partial G_3} &= 0 \iff -158.F_3 - 163.C_4 + 726.G_3 - 17.D_4 + 102.A_3 - 250.E_4 - 30.B_3{=}0\\
\frac{\partial\Sigma}{\partial D_4} &= 0 \iff -2062.F_3 + 61.C_4 - 102.G_3 + 2651.D_4 - 498.A_3 - 74.E_4 - 1134.B_3{=}0\\
\frac{\partial\Sigma}{\partial E_4} &= 0 \iff -290.F_3 - 185.C_4 - 750.G_3 - 37.D_4 - 300.A_3 + 958.E_4 - 378.B_3{=}0\\
\frac{\partial\Sigma}{\partial B_3} &= 0 \iff 30.F_3 + 15.C_4 - 10.G_3 - 63.D_4 - 18.A_3 - 42.E_4 + 118.B_3{=}0
\end{aligned}$$

With $A_3 = 220$, the above set of six equations holds six variables, and so we obtain one unique solution, according Cramer (1750, p. 59–60, App. No.1, p. 657–659). Additional beating rate equality for the fifths on B3 and altered notes leads to the solution displayed in the table below.
The diatonic fifths and major thirds have an almost equal beating rate; their beat-

Table 4. Temperament holding a minimum beating rate spread
p, q, r: interval beating rates
Δcents row: pitch deviations, in comparison with the equal temperament

Note	F3	F♯3	G3	G♯3	A3	B♭3
Pitch	175.66	184.67	196.56	207.94	220.00	234.12
Δ cents	10.33	−3.03	4.95	2.40	0.00	7.66
q	−1.71	0.29	−1.78	0.29	−2.42	0.29
p	1.70	13.09	1.75	10.83	8.63	5.20
r	−14.25	−8.04	−8.77	−16.97	−6.84	−18.91

B3	C4	C♯4	D4	E♭4	E4	F4
246.13	262.63	277.16	293.94	312.06	328.79	351.32
−5.67	6.65	−0.16	1.65	5.17	−4.41	10.33
0.29	−1.66	0.29	−1.83	0.29	−1.83	−3.43
17.55	1.99	19.49	7.67	12.19	19.58	3.41
−7.09	−15.51	−19.00	−7.07	−25.60	−7.15	−28.51

ing rate mean = 1.85 with a spread of only = 0.15. It can therefore be expected that a professional tuner is capable to set this temperament by the ear, whereby the beating rates of the diatonic fifths and major thirds is approximately 1.85, of course taking into account the required sign. The diatonic minor thirds also, have almost equal beating rate, but at a higher rate than fifths and major thirds. It can be observed, that there are six very slightly augmented fifths on B3 and on the altered notes.

6. Best diatonic cents spread

An important criticism of the above development of a circulating temperament, based on a minimal beating rate spread of diatonic intervals, could be that it does not take into account the pitch of the lower note of an interval. It might indeed be better and more logical that intervals at higher pitches would have proportionally higher beating rates. It is therefore useful to evaluate the applied purity criterion also, if based on "classically" expressed impurities in ratios or cents, rather than beating rates.
Thanks to the proportionality of impurities, the calculations become very simple. Regardless of their pitch, it is a fact that equal diatonic impurities on fifths have paired equalities of the fifths themselves, and therefore also perfect equalities within the group of involved diatonic major and minor thirds. The only condition that must be satisfied therefore, requires the equality of the absolute values of diatonic fifths and major thirds impurities. Within the diatonic C–major scale, this can be expressed by.

Equality of fifths and major thirds impurities.

$$1200 \times \log_2(fifth \times 2/3) = -1200 \times \log_2(major.third \times 4/5)$$

Combined with the condition below, that four fifths minus two octaves, end in a major third: in more simple terms this means that a major third is built, based on four fifths.

$$4 \times \log_2(fifth) - 2 = \log_2(major.third)$$

Or, after simplification.

$$\log_2(fifth) + \log_2(major.third) = -3 + \log_2(3) + \log_2(5)$$

$$4 \times \log_2(fifth) - \log_2(major.third) = 2$$

Therefore.

$\log_2(fifth) = \frac{-1+\log_2 3+\log_2 5}{5}$ hence $fifth = 1.49627787\ldots$

$\log_2(major.third) = \frac{-14+4\times\log_2 3+4\times\log_2 5}{5}$ hence $major.third = 1.25310949\ldots$

The six remaining fifths, named here as $< fifth_{alt} >$, can be equal, and must therefore satisfy with.

$fifth^6 \times fifth_{alt}^6 = 2^7$ hence $fifth_{alt} = \frac{2^{\frac{7}{6}}}{fifth} = 1.500339036\ldots$

The following scale is obtained.

Table 5. Temperament with minimum cents spread of diatonic intervals
compared to the temperament with minimum beat rate spread
p, q, r: interval cent deviations
$\Delta cents$ row: pitch deviations, in comparison with the equal temperament

Note	F3	F♯3	G3	G♯31	A3	B♭3
Cents	175.56	184.75	196.53	207.93	220.00	234.03
Beat	175.66	184.67	196.56	207.94	220.00	234.12
Δ cents	9.39	−2.35	4.69	2.35	0.00	7.04
q	−4.30	0.39	−4.30	0.39	−4.30	0.39
p	4.30	23.07	4.30	18.38	13.69	8.99
r	−22.68	−13.30	−13.30	−22.68	−8.60	−22.68

B3	C4	C♯4	D4	E♭4	E4	F4
246.27	262.69	277.18	294.06	311.97	329.18	351.13
246.13	262.63	277.16	293.94	312.06	328.79	351.32
−4.69	7.40	0.00	2.35	4.69	−2.35	9.39
0.39	−4.30	0.39	−4.30	0.39	−4.30	−8.60
23.07	4.30	23.07	8.99	13.69	18.38	8.60
−8.60	−17.99	−17.99	−8.60	−22.68	−8.60	−22.68

Both minimum spread models, the beat rate one and the ratio one, are very close to one another.

7. Impurity Spread Evaluation of Historic Temperaments

It is not easy to define a comprehensive criterion to assess temperaments Hall (1973, p. 275-277). A comprehensive temperaments criterion can only be multidemensional. The diatonic impurity spread of a number of temperaments was calculated based on the formula below, and is nothing more but a specific criterion in line with the criterion used to create a least impurity spread temperament.

$$Spread = \frac{220}{\text{diapason}}\sqrt{\frac{\sum(\text{impurity.differences})^2}{19}}$$

Comments:

- To calculate a “standardized” “diatonic impurity spread” based on interval beating rates of tonalities other than C–major, the value of the diapason must equal the pitch of the sixth of that tonality.
- The term 220/diapason should be removed for cents based calculations.

The table below was obtained. It should be easy to calculate more results if desired.

Table 6. Temperament impurity Spreads. Note: beat rate recalculated temperaments are marked by “bps”

Beating rate spread		**Spread of impurities in cents**	
Minimal beating rate spread	0.161	**Minimal cent spread**	0.000
Well Tempered Meantone	0.436	Vallotti–Tartini	0.922
Minimal cent spread	0.696	Well Tempered Meantone	1.304
Vallotti bps	0.940	**Minimal beating rate spread**	1.388
Vallotti–Tartini	0.960	Vallotti bps	1.378
Barca (Devie)	1.247	Barca (Devie)	1.840
Mercadier bps	1.268	Mercadier bps	2.792
Kirnberger III bps	1.427	Neidhardt–1	3.351
Kellner bps	1.450	Jobin	3.488
Kirnberger III	1.457	Kirnberger III	3.740
Jobin	1.504	Kirnberger III bps	3.862
Neidhardt–1	1.582	Kellner bps	3.933
Werckmeister III	2.208	Vicentio Galilei 12–TET	5.530
Werckmeister III bps	2.247	Werckmeister III	5.672
Vicentio Galilei 12–TET	2.724	Werckmeister III bps	5.954

It appears that the Well Tempered Meantone, and Vallotti, are both together by far the best ranked. See “more reading” if desired, concerning the well tempered meantone.

8. Closing thoughts

> ***General Observation***
> *It should be clear that calculations in this paper have not taken string inharmonicities into account.*
> *All reported results are approximations only. It has become clear and evident, that all* **qua'N'titative** *temperament definitions or instructions, are* **approximate indications only**. *In general, the* **qua'L'itative** *temperament definitions or instructions should therefore be considered as being the more fundamental ones.*
> *Most mid range strings have inharmonicities below one pro mille on the second partial.The auditory obtained tuned pitches in the mid range, can therefor normally be estimated to deviate some one to three cents from the theoretical and quantitative defined ones, following ratios or beating rate calculations.*
> *Keep Fourrier (1772-1837), Railsback (1938); Fletcher H. (1962), and Plomp and Levelt (1965) into mind indeed.*

The temperaments analyses based on intervals beating rates have revealed a number of temperament characteristics more in depth. It supports the hypothesis that the auditory tuning of a keyboard is mainly based on a "subjective" assessment –not measured with equipment–, of beating rate ***equalities*** of the most important diatonic intervals.
Fortunately, it appears that pitch and characteristics differences in general, between temperaments calculated on the basis of ratios, and those calculated on the basis of beatings, are insignificant. The differences are so small that those are for sure not measurable with a monochord, which is why those went unnoticed in the past, and still remain hardly noticeable today, also with most up to date pitch measuring equipment. This fortunately means, that there is no need in general, for revisions of publications based on ratios, although minor errors can sometimes be observed, due to implicite ignorance or oblivion of the schismatic comma, see for example Vallotti (1950, p. 192) and Werckmeister (1691, p. 78), –very important temperaments, after all–, and probably also some others.

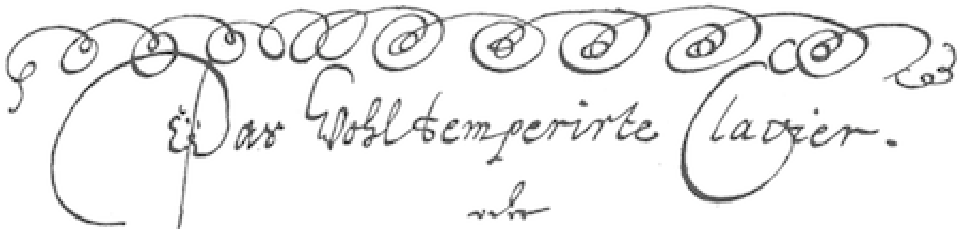
Beyond the rational within this text, the fact remains that the choice of a temperament still is and remains artistically completely free, and that if a conscious choice has to be made, this choice can therefore usually be made on artistic grounds, such as period or nature of the piece, instrument played, desired affects, composer, performing musician, etc. ...

It might become increasingly difficult to stay with normal and general acceptance that the equal temperament is the most suitable circular temperament for personal private music practice from a wide music repertoire. A good circular tuning, Vallotti par excellence, is probably the more or most suitable, ... the well tempered meantone being hardly known and accepted, and yet also somewhat more difficult to tune; ... although there might remain however a slim chance, that the well tempered meantone might one day become a desirable item for specialized didactic demonstrations?
The Vallotti temperament can be applied at will, either auditorily, –fifths beating at $-1,59$ bps.–, or on the basis of pitch measurements; the differences will hardly be measurable, ... and aurally definitely not noticeable.

More Reading suggestion : The Well Tempered Meantone

Quite some publications concerning the curls on the partition of “Das wohltemperirte Clavier” are considered controversial, but can nevertheless perhaps lead to further inspiration. In line with those numerous preceeding publications, and within the auditory tuning partition F3–F4, with a fifths distribution corresponding to a counterclock sequence on the circle of fifths, the fifths characteristics can be associated with curls characteristics, in accordance with the table below.

Note	F3	B♭3	E♭4	G♯3	C♯4	F♯3	B3	E4	A3	D4	G3	C4	F4
Fifth type	–	A	A	A	0	0	0	B	B	B	B	B	–



Original figure: Bach J.S. (1722); enhanced curls and title: Amiot (2009).

The beating rates can be obtained setting five equal fifths of type B, with an absolute value equal to the beat rate for the major third on C. This leads, unexpectedly, to an equal major third on G. The major third on F, was set equal as well, in line with meantone practice. The full set of equations was therefore.
(the first line of equations is redundant but solveable)

$$q = -p = q_{E4} = q_{A3} = q_{D4} = q_{G3} = q_{C4} = -p_{F3} = -p_{C4} = -p_{G3}$$
$$0 = q_{C\sharp 4} = q_{F\sharp 3} = q_{B}$$
$$q_{B\flat 3} = q_{E\flat 4} = q_{G\sharp 3}$$

Observation: because F3 is not included in the initial setting of C4–G3–D4–A3–E4, it might seem logical that this tuning employs the C4 note as diapason. This might be the reason why a C is associated with a curl, on the original drawing.
Following tuning information table is obtained.

Note	F3	F♯3	G3	G♯3	A3	B♭3
Pitch	175.61	184.71	196.64	207.80	220.00	234.02
Δ cents	9.85	−2.67	5.64	1.24	0.00	6.94
q	−1.17	0	−1.95	0.39	−1.95	0.39
p	1.95	12.51	1.95	12.32	8.27	5.84
r	−14.66	−8.27	−9.73	−15.39	−5.84	−18.77
B3	*C*4	*C*♯4	*D*4	*E*♭4	*E*4	*F*4
246.28	262.83	277.07	293.98	311.90	329.03	351.22
−4.62	7.96	−0.71	1.87	4.27	−3.16	9.85
0	−1.95	0	−1.95	0.39	−1.95	−2.34
16.17	1.95	19.54	7.79	13.62	17.28	3.89
−7.79	−17.51	−17.28	−7.79	−24.25	−7.79	−29.31

It can be observed that the ***natural notes are identical*** to those of the ***1/5 beat rate comma meantone*** of section 2.2.

Acknowledgements

I thank Prof. E. Amiot and A. Calvet, professional auditory tuner, lector at IRCAM, Paris, and Dr. T. Paintoux for constructive discussions and collaboration regarding the subject of this paper.
Together with G. Baroin they turned an educative video. on the Well Tempered Meantone, see site https://www.youtube.com/watch?v=lwfESoMxd8Y
Thanks finally to my daughter, advising me to think of what musicians and tuners aim at, not thinking of specific temperaments.

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