Auditory Circular Temperament Tuning

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Abstract

A musical temperament is worked out, based on a hypothetical auditory musical keyboard tuning, and is therefore unconventionally simulated by calculations based on the spread of diatonic fifths and thirds beating rates, instead of common impurity calculations with ratios, commas or cents, not including their spread : auditory tuning is based indeed on the observation and comparison of interval beating rates.

The obtained scale shows quasi equality of diatonic fifths and thirds beating rates, what leads in turn to easy auditory tuning.

A Jobin-Bach inspired temperament is worked out, and is called "Well Tempered Meantone" (WTM) in this paper, but also Vallotti and Kirnberger III are analysed.

Comparison of the spread of beating rates of temperaments shows a heading position of the here defined WTM and Vallotti. This favourable position is confirmed, also if a similar comparison is made based on a calculation of the spread of impurities expressed in proportions or cents.

The applied beating rates analysis method may open paths for alternate further analysis of temperaments, and leads to an unexpected, but somewhat assumable and historical conclusion.

Keywords

Well temperament ; circulating temperament ; meantone ; evaluation ; auditory tuning ; keyboard ; interval ; impurity ; diatonic ; Bach.

[1] Brief Circular or Well Temperament History

Musical temperament history is very well documented, for example by J. Barbour (1951), D. Devie (1990), P. Y. Asselin (1985), Di Veroli (2008) and quite some more authors.

Schlick (1511) is probably among the first written documents discussing a well temperament.

Werckmeister (1681) sets a well temperament landmark, and uses the "wohl" and "temperieren" terms also later on (1686,1689, 1698).

J. S. Bach (1722 / 1742-1744) publishes his famous masterwork "Das wohltemperirte Clavier".

Tartini (1754) comments the Vallotti temperament (Vallotti 1728).

A further brief overview of well temperament history goes necessarily at par with the question which temperament was used by J. S. Bach playing "Das wohltemperirte Clavier" on the clavichord, but there exists no historic document with precise temperament tuning instructions on this point.
It should be clear though, J. S. Bach was educated based on the meantone, but had an open musical mind, mainly focussed on the musical artistic aspects, probably accepting any good circular temperament, but with no objection against the meantone, for sure not on the organ.

Marpurg (1776, § 228, 237) believes J. S. Bach has used the equal temperament (12TET).

Today it is commonly accepted J. S. Bach dit not use the equal temperament, what has become clear, mainly because of publications of his son C. P. E. Bach (1753, §14, p. 18), and Kirnberger (1771, 1779), Bosanquet (1876, p. 29-30), Barbour (1947), Kelletat (1960), Lindley (1994).
Forkel (1802), friend with Bach's sons, testifies Kirnberger is following Bach's teaching.

Kelletat is probably at the origin of a modern quest on the temperament J. S. Bach might have used, starting with his own proposal (1966), but quite some number of posterior alternate Bach temperament proposals are published or discussed ; f. e. Kellner (1977), Billeter (1979), Barnes (1979), Lindley (1994), Sparschuh (1998), Jira (2000), Zapf (2001), Francis (2004), Lehman (2005), Allain-Dupré (2005), Jobin (2005), O'Donnell (2006), Spanyi (2006), Interbartolo/Venturino (2007), Di Veroli (2008), Amiot (2008), Broekaert (2020) ; see "Works Cited".

[2] Hypothetical Mathematical Elaboration of an idealised Auditory Tuned Circular Temperament

Among many others, a possible musical definition of Well Temperaments is proposed by Kelletat (1960 ; 1981, p. 9 ; original German text : see endnote [A]), based on Werckmeister (1681, 1689) :

<< Well temperament means a mathematical–acoustic and musical–practical organization of the tone system within the twelve steps of an octave, so that impeccable performance in all tonalities is enabled, based on the extended just intonation (natural–harmonic tone system), while striving to keep the diatonic intervals as pure as possible.
This temperament acts, while tied to given pitch ratios, as a thriftily tempered smoothing and extension of the meantone, as unequally beating half tones and as equal (equally beating) temperament. >>

The figure and listed ratios below, display the common definition of the natural harmonic system.

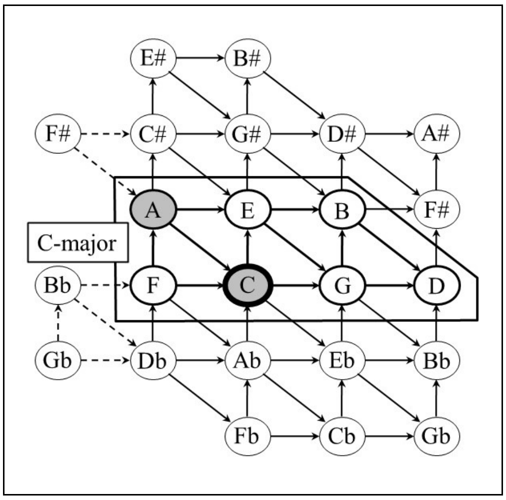

Figure 1 : Natural Harmonic System

C	D	E	F	G	A	B
1	9/8	5/4	4/3	3/2	5/3	15/8

Table 1 : Natural Harmonic Tone System

This system has a problem with the diatonic C–major purity already,: it holds six fifths –on F, C, G, D, A, E–, three major thirds –on F, C, G–, and four minor thirds –on D, A, E, B–, whereby the fifth on D (ratio : 40/27 < 3/2) and the minor third on D (ratio : 32/27 < 6/5) are not perfect or pure.

It is mathematically impossible to have all the cited intervals perfect or pure.
For a well temperament, therefore, tempering is desired already, also within the diatonic C–major.

[2.1] Mathematical Elaboration ; Applied Criterion

Auditory musical keyboard tuning is based on the evaluation of the beating rates of certain intervals, in general mainly the fifths and thirds within the F3 – E4 scale (A. Calvet).

If striving for best possible auditory diatonic purity, it is not easy, even rather impossible, to evaluate a minimum beating rate of *a GROUP of intervals*, without the aid of measuring instruments, and if no supporting beating rate data are available or calculated. A mathematical simulation of a lowest diatonic C–major beat rate, according an analytic calculation procedure similar to that of paragraph [2.2], shows a rather high spread of fifths beating rates : from – 0.77 beats/s. (F – C) to – 3.22 beats/s. (D –A).

If no instructions to obtain a given specific “Well Temperament” are given, it is on the other hand, intuitively quite easy to make an auditory evaluation on the *equality of interval beating rates*.

This way, it is possible to obtain a Well (Circulating) Temperament, based on an auditory musical keyboard tuning within the F3 – E4 scale,
not using any measuring instrument, and based on one SIMPLE, SINGLE AND UNIQUE criterion :

> Strive for a *minimum impurity spread* –i.e. a *minimum difference* of the interval beating rates– within C–Major, for following *interval GROUPS* :
> 1. The diatonic fifths and major thirds
> 2. The diatonic minor thirds
> 3. The remaining fifths

Discussion

1. Perfection or purity is desired for diatonic fifths and major thirds, but simultaneous compliance with both criteria is impossible. All fifths but one are perfect for Pythagorean tuning, but major thirds have poor quality ; on the other hand we can have eight just major thirds with meantone tuning, but at cost of a wolf fifth. In between, equal or almost equal impurities for fifths and major thirds are possible, leading to slightly flat (diminished) fifths and slightly sharp (augmented) major thirds, what might be an acceptable compromise.
2. Minor thirds impurities depend on the sum of the absolute values of fifths and major thirds impurities, and therefore those are always quite higher. It might nevertheless still be possible to keep on striving for a separate equality of those impurities.
3. The calculation of the remaining fifths does not interfere with the calculation of the diatonic notes of the diatonic C–Major tonality, therefore the least requirement consist to strive for their impurity equality.

[2.2] Calculation

The desired temperament should have a minimum for the following sum of squares of interval beating rate differences :

$$\sum differences^2 = \varphi_{F3}^2 + \varphi_{C4}^2 + \varphi_{G3}^2 + \varphi_{D4}^2 + \varphi_{A3}^2 + \varphi_{E4}^2 \quad + \quad \theta_{F3}^2 + \theta_{C4}^2 + \theta_{G3}^2$$

$$+ \vartheta_{D4}^2 + \vartheta_{A3}^2 + \vartheta_{E4}^2 + \vartheta_{B3}^2 \quad + \quad \psi_{B3}^2 + \psi_{F\#3}^2 + \psi_{C\#4}^2 + \psi_{G\#3}^2 + \psi_{Eb4}^2 + \psi_{Bb3}^2$$

based on the following definitions of beating rates differences:

Diatonic fifths: $\varphi_{Note} = \mathrm{q}_{Note} + \frac{-\mathrm{q}_{F3}-\mathrm{q}_{C4}-\mathrm{q}_{G3}-\mathrm{q}_{D4}-\mathrm{q}_{A3}-\mathrm{q}_{E4} + \mathrm{p}_{F3}+\mathrm{p}_{C4}+\mathrm{p}_{G3}}{9}$

Diatonic major thirds : $\theta_{Note} = -\mathrm{p}_{Note} + \frac{-\mathrm{q}_{F3}-\mathrm{q}_{C4}-\mathrm{q}_{G3}-\mathrm{q}_{D4}-\mathrm{q}_{A3}-\mathrm{q}_{E4} + \mathrm{p}_{F3}+\mathrm{p}_{C4}+\mathrm{p}_{G3}}{9}$

Diatonic minor thirds : $\vartheta_{Note} = r_{Note} - \frac{r_{D4}+r_{A3}+r_{E4}+r_{B3}}{4}$

Remaining fifths : $\psi_{Note} = \mathrm{q}_{Note} - \frac{\mathrm{q}_{B3}+\mathrm{q}_{F\#3}+\mathrm{q}_{C\#4}+\mathrm{q}_{G\#3}+\mathrm{q}_{Eb4}+\mathrm{q}_{Bb3}}{6}$

Whereby q_{Note}, p_{Note} and r_{Note} stand for the interval beating rate of fifths and thirds within the F3 – E4 scale, calculated according to formulas in the tables 2 to 4 below :

$q_F = 2C4 - 3F3$	$q_C = 4G3 - 3C4$	$q_G = 2D4 - 3G3$	$q_D = 4A3 - 3D4$
$q_A = 2E4 - 3A3$	$q_E = 4B3 - 3E4$	$q_B = 4F\#3 - 3B3$	$q_{F\#} = 2C\#4 - 3F\#3$
$q_{C\#} = 4G\#3 - 3C\#4$	$q_{G\#} = 2Eb4 - 3G\#3$	$q_{Eb} = 4Bb3 - 3Eb4$	$q_{Bb} = 4F3 - 3Bb3$

Table 2 : interval beating rate of fifths within the F3 – E4 scale

$p_F = 4A3 - 5F3$	$p_C = 4E4 - 5C4$	$p_G = 4B3 - 5G3$

Table 3 : interval beating rate of implicated major thirds within the F3 – E4 scale

$r_D = 10F3 - 6D4$	$r_A = 5C4 - 6A3$	$r_E = 10G3 - 6E4$	$r_B = 5D4 - 6B3$

Table 4 : interval beating rate of implicated minor thirds within the F3 – E4 scale

The above formulas are based on the first conjunction of harmonics of the concerned intervals, and some of those were already used by Kellner.
The analytical calculation of the note pitches is based on the working out of the partial derivates by the notes, set to zero, of the sum of the quadratic beating rate differences (i.e. Σdifferences^2 ; see the above formula), in accordance with an advice of Prof. E. Amiot (mathematics, musician).

It was observed, that a separate calculation of the chromatic notes of the remaining fifths Ψ_{Note} does not affect the calculation of the diatonic notes, what facilitates the calculation of the latter. Therefore, discarding the Ψ_{Note} terms that hold chromatic notes in the Σdifférences^2 formula, leads to :

$$1296 \times \sum differences^2 =$$
$$140688.F_3^2 + 71244.C_4^2 + 156816.G_3^2 + 95436.D_4^2 + 86832.A_3^2 + 68976.E_4^2 + 76464.B_3^2$$
$$-50256.F_3.C_4 - 68256.F_3.G_3 - 148464.F_3.D_4 - 11232.F_3.A_3 + 41760.F_3.E_4 + 38880.F_3.B_3$$
$$-70416.C_4.G_3 + 4392.C_4.D_4 - 54864.C_4.A_3 - 26640.C_4.E_4 + 19440.C_4.B_3$$
$$-7344.G_3.D_4 + 44064.G_3.A_3 - 108000.G_3.E_4 - 12960.G_3.B_3$$
$$-35856.D_4.A_3 - 5328.D_4.E_4 - 81648.D_4.B_3 - 43200.A_3.E_4 - 23328.A_3.B_3 - 54432.E_4.B_3$$

The partial derivatives of the above expression, set to zero, and after simplification of factors, lead to the following matrix of equations :

N	F3	C4	G3	D4	E4	B3		A3
∂Σ/∂F3	1954	–349	–474	–1031	290	270	=	78
∂Σ/∂C4	–698	1979	–978	61	–370	270	=	762
∂Σ/∂G3	–158	–163	726	–17	–250	–30	=	–102
∂Σ/∂D4	–2062	61	–102	2651	–74	–1134	=	498
∂Σ/∂E4	290	–185	–750	–37	958	–378	=	300
∂Σ/∂B3	30	15	–10	–63	–42	118	=	18

Table 5 : Set of equations, calculating the diatonic note pitches, for best possible equality of interval beating rates

Solution of the equations table 5, combined with a simple additional calculation of chromatic notes, based on equal Ψ_{Note} values, leads to the following scale :

	F3	F#3	G3	G#3	A3	Bb3	B3	C4	C#4	D4	Eb4	E4
f_{Note}	175.66	184.67	196.56	207.94	220.00	234.12	246.13	262.63	277.16	293.94	312.06	328.79
q_{Note}	–1.71	0.29	–1.79	0.29	–2.42	0.29	0.29	–1.66	0.29	–1.83	0.29	–1.83
p_{Note}	1.70	13.09	1.75	10.83	8.63	5.20	17.55	1.99	19.49	7.67	12.19	19.58
r_{Note}	–14.25	–8.04	–8.77	–16.97	–6.84	–18.91	–7.09	–15.51	–19.00	–7.07	–25.60	–7.15
cents	10.33	-3.03	4.95	2.40	0.00	7.66	-5.67	6.65	-0.16	1.65	5.17	-4.41

Table 6 : "Ideal" beating scale characterised by a best possible equality of the following groups of intervals :
diatonic fifths and major thirds ; diatonic minor thirds ; remaining fifths

A quasi beat rate equality of quite a number of intervals can be observed. It can be expected a professional auditory tuner should be able to achieve a scale, with fifths beating rates around 1.85 –the mean of the diatonic fifths and major thirds at table 6, with a spread of only : 0.05–.

[3] A practical pragmatic application of possible "optimal" auditory tuning

[3.1] A Well Tempered Meantone (WTM)

Jobin proposed a temperament, derived from the meantone, containing five equal diatonic fifths, –C, G, D, A, E–, and two pure major thirds, –C, G–. His proposal is based on classical impurity calculations in cents, and is inspired by curls on a partition of "Das wohltemperirte Clavier" of J. S. Bach.

Inspired by the above and the procedure of par. 2.2, a scale can be calculated for best possible beating rate equality of the above mentioned intervals. A perfect equality is *unexpectedly* achieved.
Because of the unexpected equalities, and such as with the 1/5 c. meantone, a third equal major third can be added on F, instead of a perfect fifth, as was done by Jobin. Further on, such as for the Jobin proposal, we can have perfect fifths on B3, F#3 and C#4, and fifths with equal beating rate on G#3, E*b*4, B*b*3. Because of the analytically discovered unexpected equality, a more simple calculation is possible, by setting :

$$q_{C4} = q_{G3} = q_{D4} = q_{A3} = q_{E4} = -p_{F3} = -p_{C4} = -p_{G3}$$

$$q_{B3} = q_{F\#3} = q_{C\#4} = 0 \qquad q_{G\#3} = q_{Eb4} = q_{Bb3}$$

The first set of seven equations is redundant ; it contains only six variables, the notes F, C, G, D, E, B, but can be solved ; ONE requirement can be discarded : either that of q_{C4} or p_{C4} or p_{G3} or q_{E4} .
The solution for the diatonic notes is very simple, and could even look familiar to Baroque musicians and musicologists, with proportions:

$$-q_{Note} = p_{Note} = \frac{A3}{113} = \frac{5F3}{451} = \frac{C4}{135} = \frac{G3}{101} = \frac{D4}{151} = \frac{E4}{169} = \frac{2B3}{253}$$

The following scale is obtained :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.61	184.71	196.64	207.80	220.00	234.02	246.28	262.83	277.07	293.98	311.90	329.03
q_{Note}	– 1.17	0.00	– 1.95	0.39	– 1.95	0.39	0.00	– 1.95	0.00	– 1.95	0.39	– 1.95
p_{Note}	1.95	12.51	1.95	12.32	8.27	5.84	16.17	1.95	19.54	7.79	13.62	17.28
r_{Note}	– 14.66	– 8.27	– 9.73	– 15.39	– 5.84	– 18.77	– 7.79	– 17.51	– 17.28	– 7.79	– 24.25	– 7.79
cents	9.85	– 2.67	5.64	1.24	0.00	6.94	– 4.62	7.96	– 0.71	1.87	4.27	– 3.16

Table 7 : Well Tempered Meantone ; see grey cells, for important equal beating rates
The cents display the deviations from the equal temperament (12TET)

Observation :
The WTM could be referred to as some kind of "equal" temperament : it has quite a number of intervals with *equal beating rate* ("gleichschwebend"). This "equal" differs with the "equal" of the "Equal Temperament" (12 TET) that has *proportional equality* of twelve fifths ("gleichstufig"). See also footnote [B].

[3.1.1] Main Characteristics

Auditory tuning of this temperament is easy, because of its beating rate equalities, see table 7.
A video, demonstrating the ease of the auditory tuning, is displayed on the internet :

https://www.youtube.com/watch?v=lwfESoMxd8Y

This video was compiled by PhD. Baroin G., in cooperation with professor Amiot E., and Calvet A., professional auditory tuner.

The main interval characteristics are very simple to memorise, and are displayed in figure 2 below :

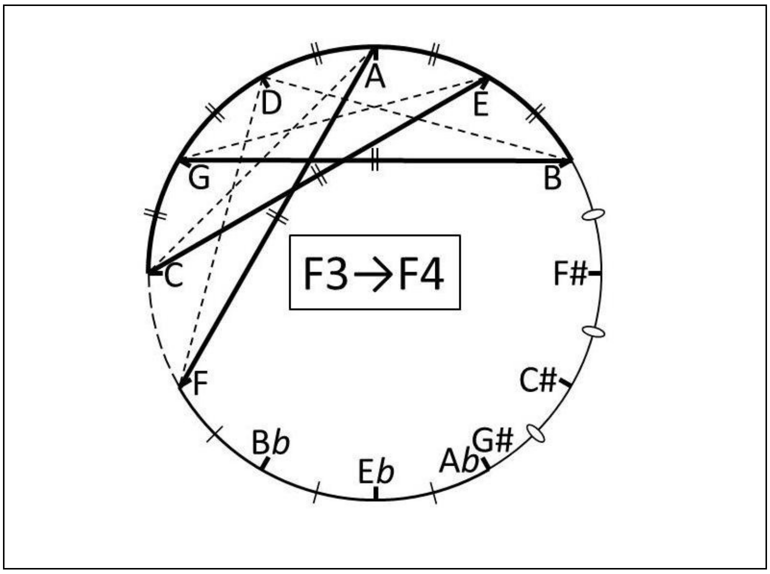

Fig 2: Well Tempered Meantone ; Circle of fifths
Intervals marked by " = " : – (fifths) or + (major thirds) 1.95 bps
Fifths marked by " 0 " : perfect
Fifths marked by " | " : + 0.39 bps (plus (!) 0.39)
Fifth on " F " ; not determined (~ – 1.17 bps)

Why to name this scale a "Well Tempered Meantone" ?

As with meantone, the auditory tuning is initiated by setting four equally flat fifths – on A, D, G, C–, but so that it leads to an equally sharp major third on C, instead of a pure one.

As with meantone, the auditory tuning is continued, by setting two more and equal major thirds, on G and on F ; unexpectedly leading to an additional equally flat fifth on E ; the set of obtained equalities is similar to those of the 1/5 comma meantone.

Differing from the meantone, the auditory tuning is not finalised by setting more similar major thirds, but by setting three perfect fifths on B, F#, C#, and the remaining fifths almost perfect, leading to well tempering.

[3.1.2] Musical and Historic Factors

[3.1.2.1] WTM Fifths :

The fifths beating rates match very well the curls characteristics within a figure on top of a partition of "Das wohltemperirte Clavier" ; see the figure 3 below.

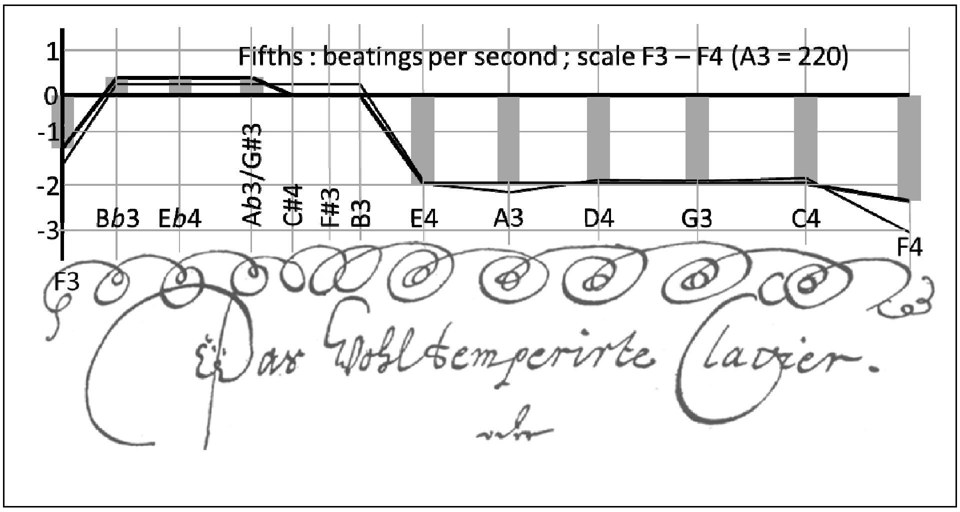

Figure 3 : "Well Tempered Meantone" (table 7) : fifths impurities in beatings per second : see the bold line / grey bars
Thin line : fifths impurities in beatings per second, of the scale with "optimal" purity (table 6)

[3.1.2.2] Semitones

Kelletat emphasizes the importance of natural and chromatic semitones sizes, all over his four books (1960, 1981. 1982, 1994). Some similarity with meantone semi tones is required, and to his opinion, the Kirnberger III semi tones are satisfactory for Bach's music (1960, 1981), Viennese classical music (1982) and Schubert's "Liedern" (1994).
WTM semitone characteristics are graphically on display in fig. 4.

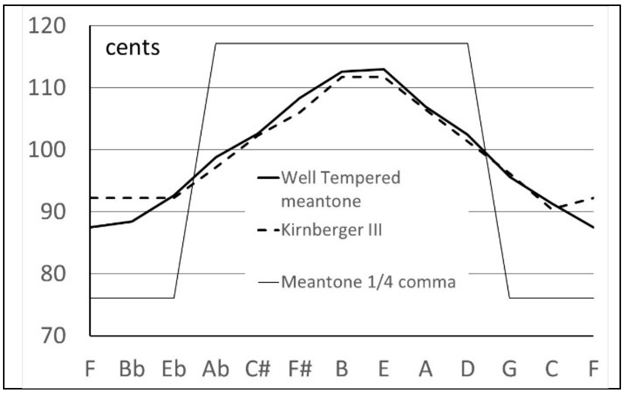

Fig. 4: Well Tempered Meantone ; Semi tone Characteristics

[3.1.2.3] WTM Triton :

The WTM has a remarkable F–B triton. The expression 7F – 5B , calculating a possible beating rate for the F–B triton interval calculates a beating rate of only – 2.14 bps. The meantone has more better quality tritons, including a comparable F–B triton.

	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#
WTM	-15.3	-16.7	-7.0	-2.1	-7.3	-8.9	-20.1	-19.5	-37.0	-32.1	-21.2	-26.9	-15.3
Meantone	-32.0	3.9	2.9	2.2	3.2	2.4	3.6	-34.2	-51.1	-38.2	-28.6	-42.7	-32.0
12TET	-14.8	-22.1	-16.6	-12.4	-18.6	-13.9	-20.9	-15.6	-23.4	-17.5	-13.1	-19.7	-14.8
Kirnberger III	-15.1	-16.4	-7.3	-5.5	-8.2	-9.3	-20.5	-19.7	-36.6	-27.4	-20.6	-26.3	-15.1

Table 8 : WTM Triton beating rates

[3.2] Vallotti

The almost even diatonic fifths distribution of table 6, might, for example, also lead to the “classic” Vallotti tuning, holding six equally flat fifths.
Full details on the Vallotti temperament are available today (Vallotti, 1950, Chap. IV, pp. 192-201). See also Barbieri (1987), Di Veroli (2008).

It is remarkable, Vallotti too strives for high purity of the diatonic C–major tonality, such as for the above discussed WTM, and he observes a deviation of one syntonic comma on the fifth D – A (see also the ratios displayed in table 1). Because of fifths importance, he argues diatonic purity can at first be improved by :

- a distributed diminution of the six diatonic fifths by 1/6 of a syntonic comma
- 6 perfect fifths on B to Bb

The above conditions contain a mathematical contradiction : an error of one schismatic comma ; the difference between the Pythagorean comma ($3^{12}/2^{19}$) and the syntonic one (80/81). *This residual schismatic comma is* NOT *discussed, actually ignored*.

Vallotti does not comment HOW *to tune the flat fifths*. Monochord tuning requires exact figures on required proportions, and those are not worked out. The unveiled schismatic comma shortcoming opens three possible temperaments :

- Version 1 : six diatonic fifths flat by 1/6-th of a syntonic comma, and 1/6-th of a schismatic comma spread over the remaining fifths
- Version 2 : six diatonic fifths flat by 1/6-th of a syntonic comma, five remaining perfect fifths, and one fifth with the schismatic comma, probably the fifth on Bb, as last one remaining to be controlled or tuned
- Version 3 : six diatonic fifths flat by 1/6-th of a Pythagorean comma, and six perfect fifths,

The differing fifths cord lengths (in mm) on a monochord having a one meter chord, and the corresponding deviations in cents, are displayed in the table below

	Diatonic fifths	Chromatic Fifths ; except Bb	Fifth on Bb
Version 1	668.05 mm – 3.58 cents	666.79 mm – 0.36 cents	666.79 mm – 0.36 cents
Version 2	668.05 mm – 3.58 cents	666.67 mm 0 cents	667.42 mm – 1.95 cents
Version 3	668.17 mm – 3.91 cents	666.67 mm 0 cents	666.67 mm 0 cents

Table 9 : Vallotti fifths characteristics

Observations :

- Achievable monochord measurement precision : the monochord precision is limited because of chord inharmonicity. The monochord precision can, for example, be compared to that measured in the middle octave of an upright Bruckner piano, on sound samples longer than 50 cycles of the first harmonic. This reveals 0.2 % deviations or more on the third harmonic, corresponding with 3.5 cents, this is 1.3 mm on a 1 m. monochord.

Further observations :

- Differences of diatonic fifths are too small to be measureable (0.13 mm ; 0.33 cents)
- Differences of chromatic fifths are too small to be measureable (0.13 mm ; 0.36 cents), except for Bb in version 2 ; therefore version 2 can be excluded, as there is no comment on this fifth in Vallotti's text
- Remaining versions 1 and 3 : no choice is possible ; too small differences, and Vallotti's arguments contain a mathematic contradiction

Remains the tuning procedure.
Analysis of required cord lengths for the diatonic fifths reveals a difference of only 1.4 mm from the perfect position, for a chord of 1m. This has to be set six times, and should achieve an overall error of the diatonic fifths lower than the schismatic comma (1.85 cents, or 0.75 mm), in order to lead to an acceptable result. Therefore, this procedure is probably very hard to apply because of evident lack of monochord precision.
The monochord could have been applied for measurements "a posteriori". Those may have lead to Vallotti's theory, because of unperceived or unsuspected measuring inaccuracies at his time.

All the above might support an hypothesis that auditory tuning might have been applied, the normal and current keyboard tuning procedure at Vallotti's time, possibly based on even auditory distribution of the comma. Corresponding beating rates and cord lengths are worked out in the table below.

	F	F#	G	G#	A	Bb	B	C	C#	D	Eb	E
Pitch	175,49	184,88	196,44	207,99	220,00	233,99	246,51	262,45	277,32	293,86	311,99	329,21
pNOTE	-1,59	0,00	-1,59	0,00	-1,59	0,00	0,00	-1,59	0,00	-1,59	0,00	-1,59
pNOTE	2,54	11,56	3,85	9,83	9,29	5,50	15,41	4,60	17,33	9,74	11,56	17,90
Chord mm	668.68	666.67	668.47	666.67	668.27	666.67	666.67	668.01	666.67	667.87	666.67	667.77
5-th cents	-5.23	0.00	-4.67	0.00	-4.17	0.00	0.00	-3.49	0.00	-3.12	0.00	-2.78

Table 10 : Auditory tuned Vallotti / 5-th cents : cents deviations from perfect fifths

It is clear the differences in diatonic fifths chord lengths are minimal : a maximum deviation from the mean value of only 0.94 mm. ; this is of the same order of magnitude as piano chord inharmonicity. The auditory tuning should on the other hand be rather easy, because of low and even harmonic beat rate of all C-major diatonic fifths, the perfection of the chromatic fifths, all this also combined with possible control points on its major thirds, including also the major third on Bb.

[3.3] Kirnberger III

The Kirnberger III temperament is known from a 1779 letter to Forkel (Kelletat 1982, p. 140). Kirnberger III also, is a temperament striving for best possible diatonic purity, and has equal flat fifths on C, G, D, A, so that a pure major third on C is obtained. Further on all fifths are perfect, except the one F#.

Kelletat (1982, table 12) suggests a version with almost equal beating rates, he names it Kirnberger III ungleich (not(!) equal). The division in cents of the major third on C is not equal indeed, but Kelletat reports almost equal beating rates for G, D, A, and approximately half the rate one on C.
In line with preceding approaches for the proposed WTM and Vallotti, the Kirnberger III

temperament can, as a variant, be tuned by setting a pure third on C based on four flat fifths, on C, G, D, A, holding an equal beating rate in the F3-E4 scale. This tuning can start on A as well as C.

F	F#	G	G#	A	Bb	B	C	C#	D	Eb	E
175.41	185.00	196.78	297.89	220.00	233.88	246.67	263.12	277.19	294.07	311.84	328.89
0.00	-0.63	-2.21	0.00	-2.21	0.00	0.00	-2.21	0.00	-2.21	0.00	0.00
2.95	10.51	2.76	12.99	8.77	6.88	14.01	0.00	17.32	9.67	15.07	18.68

Table 11 : Auditory tuned Kirnberger III

Kirnberger III is, just as the WTM and Vallotti, among the easiest temperaments for auditory tuning .

[3.4] More Ancient Temperaments

The monochord was not useful for precise tuning according Vallotti, and monochord measuring results had unperceived inaccuracies. Arguments concerning those shortcomings, are probably valuable for many historic temperaments, developed preceding modern pitch measuring technologies.

Therefore, the often reported even distribution of fractions of comma's are probably possible misinterpretations of measurement results on installed temperaments, whereby it is often also not stated how the tuning was performed.
Many temperaments could or should therefore be reanalysed based on possible auditory tuning observations of the discussed intervals.
The thus obtained historic temperaments, possibly holding even beating rate distribution of commas, are always very close to those obtained by calculating proportions. Sometimes discussion of the theories advanced by their authors might possibly be found erroneous, such as it is the case for Vallotti. Therefore, there remains a large open field for new profound temperament analysis.

[4] Comparison with other Well Temperaments

[4.1] A "Classic" ideal scale, based on interval proportions

An important critic on the preceding paragraphs [2 and 3] might hold the fact that those treat interval beating rate equality, not taking the level of notes pitches into account ; higher pitches might indeed better hold proportionally higher interval beating rates. It makes sense, therefore, to evaluate the applied purity criterion also, if based on "classically" expressed impurities in proportions or cents, instead of interval beating rates.

Because of the proportionality of impurities, the calculations become very simple. Independently on their pitches, we have that equal diatonic fifths impurity differences leads to fifths equalities, and hence, in turn also to perfect equality within the group of the diatonic major thirds and the group of diatonic minor thirds. The only condition that has to be set therefore, requires the equality of the absolute impurity values of diatonic fifths and major thirds. This can be expressed by :

Equality of impurities:

$$1200log_2(fifth \times 2/3) = -1200log_2(majorthird \times 4/5)$$

combined with the condition below, that four fifths minus two octaves lead to a major third :

$$4log_2(fifth) - 2 = log_2(majorthird)$$

Or, after simplification :

$$log_2(fifth) + log_2(majorthird) = -3 + log_2 3 + log_2 5$$
$$4log_2(fifth) - log_2(majorthird) = 2$$

Hence :

$$log_2(fifth) = \frac{-1 + log_2 3 + log_2 5}{5} \qquad fifth = 1.49627787 \ldots$$

$$log_2(majorthird) = \frac{-14 + 4log_2 3 + 4log_2 5}{5} \qquad majorthird = 1.253109491 \ldots$$

The remaining six fifths, called $fifth_{alt}$ here, can be equal, and have therefore to satisfy :

$$fifth^6 \times fifth_{alt}^6 = 2^7 \quad \text{hence} \quad fifth_{alt} = \frac{2^{\frac{7}{6}}}{fifth} = 1.500339036 \ldots$$

Obtained scale : (the impurities q, p and r of fifths and thirds are expressed in cents)

	C4	C#4	D4	Eb4	E4	F4	F#4	G4	G#4	A4	Bb4	B3
f_{Note}	262.69	277.18	294.06	311.97	329.18	351.13	369.49	393.06	415.87	440.00	468.06	492.55
q_{Note}	–4.30	0.39	–4.30	0.39	–4.30	–4.30	0.39	–4.30	0.39	–4.30	0.39	0.39
p_{Note}	4.30	23.07	8.99	13.69	18.38	4.30	23.07	4.30	18.38	13.69	8.99	23.07
r_{Note}	–17.99	–17.99	–8.60	–22.68	–8.60	–22.68	–13.30	–13.30	–22.68	–8.60	–22.68	–8.60
cents	7.0	0.0	2.3	4.7	-2.3	9.4	-2.3	4.7	2.3	0.0	7.0	-4.7

Table 9 : “Ideal” proportional scale with proportional interval deviation equality of the diatonic fifths and thirds
Notice : minor thirds impurities equal the sum of absolute values of fifths and major thirds impurities

[4.2] Impurity Evaluations

Although it is not easy to define a comprehensive algorithm for well temperament evaluation (Hall D. ; p. 275-277), a number of well temperaments are compared, based on the calculation of their spread of impurity differences ; this criterion was indeed used here to define well tempered scales. This impurity spread calculation can be made based on interval beating rates, but also based on interval cents deviations. The spread is calculated based on the formula below, applied within the F3 – E4 scale:

$$\text{Diatonic purity spread} = \frac{220}{Diapason}\sqrt{\frac{\sum \text{impurity differences}^2}{19}}$$

Notes : For the “beating rate” calculation of the “Diatonic purity spread”, of tonalities other than C–major, the value of the diapason has to be equal to the pitch of the sixth of that tonality.
The term 220/*Diapason* has to be discarded for cents-based calculations.
Table 12, see endnote [C], displays obtained C–major diatonic impurity spreads, of a collection of circulating temperaments.

Observations (see table 12 at endnote [C]) :

- Temperaments marked by “(bach)” have at times, ever been claimed as applied by J. S. Bach.
- Temperaments marked by “bps”, are recalculated temperaments, on the basis of even division of the comma by setting equal interval beating rates, instead of equal interval cents or proportions.
- The WTM stands on top of the table, and is joined, in this position, by the Vallotti temperament ; see also their graphical comparison in the figure 5 below.
  A Valotti tuned keyboard might therefore, today, be the best approximate indication on how the WTC of J. S. Bach could have sound at Bach’s time, ... if it were the WTM should ever have been used by Bach indeed.

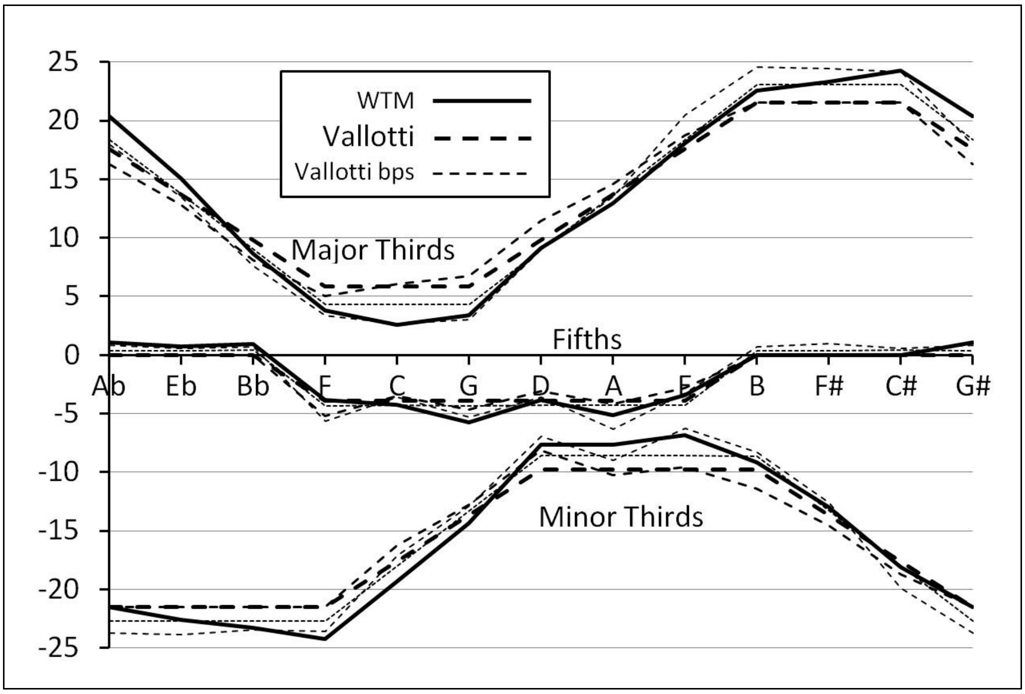

Fig 5 : Interval Impurities (cents)

[4] Conclusion

When discussing musical temperaments :

Even interval beating rates import for well temperaments.
Correct interpretation of the “equal” term imports : what are the concerned intervals, what kind of equality is discussed (interval ratio <–> beating rate, or also : “gleichstufig” <–> “gleichschwebend”) ?

A mathematical analysis of interval beating rates can be revealing for some historic temperaments : many of those were indeed based on reported auditory tuning recommendations only, but were often calculated afterwards, based on proportions rather than the for auditory tuned temperaments more appropriate and estimated interval beating rates.

The Well Tempered Meantone :

Has quite a number of important intervals with equal beating rate (table 7).

Is easy to tune auditory (paragraph 3, 3.1, table 7). Auditory tuning is probably easiest for Kirnberger III –only four equally flat fifths controlled by a just major third, followed by seven perfect fifths–, followed by Vallotti.

Stands on top of well temperaments, if sorted for low auditory impurity spread of the diatonic C–Major tonality (table 10).

Is joined, in this position, by the Vallotti temperament (table 10).

Has acceptable semitones and triton characteristics (paragraphs 3.2.1, 3.2.2).

Has fifths characteristics that match very well with the characteristics of curls on the partition of “Das wohltemperirte Clavier” of J. S. Bach (paragraph 3, figure 3).

Acknowledgment

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Their open attitude allowed for quite intense exchange of ideas, that enabled further understanding and evolvement of insights in musical temperament and tuning matters, enabling the development of the ideas and concepts expressed in this paper.

Thanks to my daughter Hilde : she drew my attention to investigate on what musicians want (diatonic interval purity) and not on what might be someone's preferred musical temperament.

WORKS CITED

Allain-Dupré P. 2005 : "Justesses et Tempéraments" (academia.edu) variation of Lehman's proposal.
Amiot E. 2008 : "Discrete Fourier Transform and Bach's Good Temperament" ; MTO, Volume 15, Number 2, June 2009
Asselin P. Y. 1985 : " Musique et Tempérament"
Bach C. P. E. 1753 : "Versuch über die wahre Art des Clavier zu spielen, mit Exemplen und achtzen Probe=Stücken in sechs Sonaten".
Barbieri P. 1987 : Acustica, Accordatura e Temperamento nell' illuminisimo veneto.
Barbour J. 1947 : "Bach and the Art of Temperament".
Barbour J. 1951 : "Tuning and temperament : a historical survey".
Barnes J. 1979 : "Bach's Keyboard Temperament: Internal Evidence from the Well-tempered clavier" ; Early Music, Volume 7, Issue 2, April 1979, Pages 236–249,
Billeter B. 1979 : "Anweisung zum Stimmen von Tasteninstrumenten in verschiedenen Temperaturen" (ISBN 3-87537-160-7).
Billeter B. 2008 : "Zur 'Wohltemperirten' Stimmung von Johann Sebastian Bach: Wie hat Bach seine Cembali gestimmt?" ; (Ars Organi Zeitschrift, 2008-3, p. 18-21)
Bosanquet R. 1876 : "An elementary treatise on Musical Intervals and Temperament". Macmillan & Co. 1876
Broekaert J. 2020 : "Le Mésotonique Tempéré de Bach" (Pianistik No 111, dec. 2020, p. 4-19)
Calvet André, 2020 : "Le Clavier Bien Obtempéré" (ISBN 978-2-9541401-3-1)
Devie D. 1990 : "Le Tempérament Musical"
Di Veroli C. 2008 : "Unequal Temperaments: Theory, History and Practice" (e-book) The Viola da Gamba Society Journal, Volume Four, (2010)
Forkel J. 1802 : "Über Johan Sebastian Bach's Leben, Kunst und Kunstwerke". Leipzig, Hoffmeister und Kühnel. (Bureau de Musique.) 1802.
Francis J. C. 2004 : "The Keyboard Temperament of J. S. Bach" (Eunomios).
Francis J. C. 2005-2 : "The Esoteric Keyboard Temperaments of J. S. Bach" (Eunomios)
Francis J. C. 2005-7 : "Das Wohltemperirte Clavier, Pitch, Tuning and Temperament Design" (Eunomios)
Fritz B. 1757 : "Anweisung, wie man Claviere, Clavecins, und Orglen nach einer mechanischen Art, in alle zwölf Tönen gleich und rein stimme könne, daß aus solchen allen sowol dur als moll wohlklingend zu zpielen sein. Leipzig 1757.
Hall D., 1973 : "The Objective Measurement of Goodness-of-Fit for Tunings and Temperaments", Journal of Music Theory, Vol. 17. No. 2. (Autumn 1973)
Interbartolo G., Venturino P 2007 : Bach 1722 "Il temperamento de Dio". {ISBN A000068628).
Jedrzejewski F. 2002 : Mathématiques des systèmes acoustiques : Tempéraments et modèles contemporains.
Jira M. 2000 : "Musikalische Temperaturen und Musikalischer Satz in der Klaviermusik von J. S. Bach" (2000, Hans Schneider – Tutzing}. ISBN 3-79521-004-6).
Jobin E. 2005 : "BACH et le Clavier bien Tempéré" ; (website of "Clavecin en France").
Jorgesen O. 1991 : "Tuning ; Containing the Perfection of Eighteenth-century Temperament, the Lost Art of Nineteenth-century Temperament, and the Science of Equal Temperament, Complete with Instructions for Aural and Electronic Tuning".

Kelletat H. 1960 : “Zur musikalischen Temperatur”. Kassel, Oncken 1960
Kelletat H. 1966 : “Ein Beitrag zur musikalischen Temperatur der Musikinstrumente vom Mittelalter bis zur Gegenwart“. Reutlingen,
Kelletat H. 1981 : “Zur musikalischen Temperatur”; Band I. Johann Sebastian Bach und seine Zeit“. ISBN 3–87537 156–9
Kelletat H. 1982 : “Zur musikalischen Temperatur” ; Band II. Wiener Klassik“. ISBN 3–87537 187–9
Kelletat H. 1994 : “Zur musikalischen Temperatur” ; Band III. Franz Schubert”. ISBN 978–3–87537–239–5
Kellner H. 1977 : “Eine Rekonstruktion der wohltemperierten Stimmung von Johann Sebastian Bach“. Das Musikinstr. 26, 1977, 34-35
Kirnberger J. 1771 : “Die Kunst des reinen Satzes in der Musik”, ISBN 3–487–01875–6
Kirnberger J. 1782 : “Gedanken über die verschiedenen Lehrarten in der Komposition”. Berlin: Georg Jacob Decker.
Lehman B. 2005 : “Bach’s extraordinary temperament: our Rosetta Stone – 1 ; – 2” (Early Music, vol. 33, No 1, feb 2005, p.3-23 ; vol. 33, No 2, may 2005 p. 211-231).
Reaction : a number of letters are addressed to “Early Music” : Jencka D. (2005–8, p. 545) ; Maunder R. (2005–8, p. 545–546) ; Mobbs K., MacKenzie A. (2005–8, p. 546–547),
Lehman B. (2006–4) : “Bach’s Art of Temperament” ; (Website of Microstick)
Lindley M. 1993 : “Mathematical Models of Musical Scales”. (Verlag für systemathische Musikwissenschaft GmbH, Bonn).
Lindley M. 1994 : "A Quest for Bach's Ideal Style of Organ Temperament" (M. Lustig, ed., Stimmungen im 17. und 18. Jahrhundert, Michaelstein, 1997).
Lindley M., Ortgies I. (2006-11) : “Bach style keyboard tuning” ; (Early Music, 2006-11, p. 613-623).
Marpurg F. 1776 : “Versuch über die musikalische Temperatur”. (ISBN 0-36408-671-8).
O’Donnell J. 2006 : : “Bach's temperament, Occam's razor, and the Neidhardt factor” (Early Music, 2006–11, p. 625-633)
Rossi L, 1666 : “Sistema musico, ouero Musica speculativa doue SI spiegano i più celebri sistemi di tutti i tre generi”. In Perugia, nella Stampa Episcopale, per Angelo Laurenzi, 1666 - in 4°. Sei carte non numerate in principio e fac. 179. 1666.
Sauveur J. 1701 : “Principes d'acoustique et de musique, ou système général des intervalles des sons”. Inséré dans les mémoires de 1701. De l’Academie Royale des Sciences.
Schlick A. 1511 : ‘Spiegel der Orgelmacher und Organisten, allen Stiften und Kirchen, so Orgeln halten oder machen lassen, hochnützlich’, Speyer 1511.
Spanyi M. 2006 : "Kirnberger's Temperament and its Use in Today's Musical Praxis" (Clavichord international – 11 (2007-5), 1, Seite 15-22).
Sparschuh A. 1999 : “Stimm– Arithmetic des wohltemperierten Klaviers von J. S. Bach” (Deutsche Mathematiker Vereinigung, Jahrestagung 1999, Mainz, S. 154–155).
Tartini G. 1754 : “Trattato di musica secondo la vera scienza dell’ armonia” (p. 100).
Vallotti F. 1779 (1728) : “Della scienza teorica e pratica della moderna musica” (book 1).
Vallotti F. 1950 : "Trattato della Moderna"Musica"
Werckmeister A. 1681 : “Orgelprobe”. Theodorus Phil. Calvisius, Buchhändl in Quedlinburg
Werckmeister A. 1686 : “Musicae Hodegus Curiosus”. (ISBN 9783487040806)
Werckmeister A. 1689 : “Musicalische Temperatur”. Theodorus Phil. Calvisius, Buchhändl in Quedlinburg
Werckmeister A. 1698 : “Orgelprobe”. Leipzig, bei Johann Michael Teubner
Zapf M. 2001 : "Handing down the Tradition: The survival of Bach's Finger Technique in an Obscure Nineteeht-Century Clavier Tutor". (De Clavicoridio V, sept. 2001, p. 39-44)

[A] Wohltemperierung heißt mathematisch=akustische und praktisch=musikalische Einrichtung von Tonmaterial innerhalb der zwölfstufigen Oktavskala zum einwandfreien Gebrauch in allen Tonarten auf der Grundlage des natürlich= harmonischen Systems mit dem Bestreben möglichster Reinerhaltung der diatonischen Intervalle.
Sie tritt auf als proportionsgebundene, sparsam temperierende Lockerung und Dehnung des mitteltönigen Systems, als ungleichschwebende Semitonik und als gleichschwebende Temperatur.

[B] The Equal Temperament (12TET)
B. Fritz has proposed a temperament whereby all fifths should be flat. He describes an auditory procedure, starting at the F note, surrounding the circle of fifths so that it is closed properly by tuning slightly flat fifths ; it is written how flat fifths should be obtained but no exact figures are given, … some assume fifths should be equal. It is clear that with

equal fifths, there are twelve possible and differing scales, depending on the initial note on which the scale is built. The table below displays the twelve possible beating rates (in beats/sec).

Initial note	F3	F#3	G3	G#3	A3	Bb3	B3	C4	C#4	D4	Eb4	E4
Beating rate	– 0.80	– 0.84	– 0.89	– 0.95	– 1.00	– 1.06	– 1.13	– 1.19	– 1.26	– 1.34	– 1.42	– 1.50

Comparison of those twelve scales with the 12TET reveals that the mean by note of their pitches equals the pitches of the 12TET, with a maximum deviation of only 0.9 cent, and a RMS deviation of only 0.4 cent. Those differences lie well within the normal achievable auditory tuning precision, what is confirmed by the fact that auditory tuning of the equal temperament is commonly evaluated as very difficult. It can therefore be said that all the equal temperaments are musically equivalent.

[C]

Impurity Spread of Well Temperaments			
Interval beating rate differences		**Interval cents differences**	
Ideal Beating equality	0.161	**Ideal Proportional Equality**	0.000
Well Tempered meantone	0.436	Vallotti - Tartini	0.922
Ideal Proportional Equality	0.696	**Well Tempered meantone**	1.304
Vallotti bps	0.940	**Ideal Beating equality**	1.388
Vallotti - Tartini	0.960	Vallotti bps	1.738
Venturino 1/4 1/19 (bach)	1.175	Barca (Devie)	1.840
Venturino 1/4 1/12 corr (bach)	1.208	Lindley 1994 Michaelstein (lehm 1)	2.178
Barca (Devie)	1.247	Young 1800	2.460
Mercadier bps	1.268	Lambert 1774	2.564
Young 1800	1.270	Mercadier	2.591
Lindley 1994 Michaelstein (lehm 1)	1.275	Mercadier bps	2.792
Mercadier	1.305	Sievers	2.798
Maunder b (bach)	1.360	Maunder b (bach)	3.004
Sievers	1.360	Jencka (bach)	3.040
Jencka (bach)	1.366	DI Veroli WTC opt (bach)	3.306
Barnes	1.399	Barnes	3.307
Lehman_1_6_Pyth	1.403	Lehman_1_6_Pyth	3.307
Kirnberger III bps	1.427	Neidhardt-1	3.351
Venturino 1/4 1/12 (bach)	1.445	Young / Van Biezen	3.367
Kellner bps	1.450	Lindley 1994 Average Neidhardt (lehm 2)	3.376
Kirnberger III	1.457	Neidhardt 1 bps	3.390
Lehman bps	1.460	Venturino 1/4 1/19 (bach)	3.488
Neidhardt 1 bps	1.461	Jobin	3.488
Lambert 1774	1.461	Asselin	3.510
Jobin	1.504	Barca (Asselin)	3.592
Kellner	1.521	Lehman bps	3.613
Legros (2 R.T.)	1.546	Weingarten / Gabler	3.620
DI Veroli WTC opt (bach)	1.553	Legros (2 R.T.)	3.670
Neidhardt-1	1.582	Francis 2005 1/14 PC (bach)	3.704
Weingarten / Gabler	1.592	Maunder c (bach)	3.704
Lindley 1994 Average Neidhardt (lehm 2)	1.649	Venturino 1/4 1/12 corr (bach)	3.709
d'Alembert / Rousseau	1.681	Kirnberger III	3.740
Young / Van Biezen	1.685	Kirnberger III bps	3.862
Francis 2005 1/14 PC (bach)	1.737	Kellner	3.888
Jira geschlossen 2 (bach)	1.739	Kellner bps	3.933
Barca (Asselin)	1.751	Jira geschlossen 2 (bach)	3.964
de Bethisy	1.788	Sparschuh 1999 (bach)	4.071
Jira offen 1 (bach)	1.803	Francis 2005 EB (bach)	4.126
Kirnberger III ungleich	1.835	Neidhardt-2	4.169
Asselin	1.838	Sorge1744	4.172
Sparschuh 1999 (bach)	1.847	d'Alembert / Rousseau	4.259
Neidhardt 4 bps	1.905	Kirnberger III ungleich	4.344
Neidhardt-2	1.933	Neidhardt 2 bps	4.355
Neidhardt 2 bps	1.968	Sorge 1744 bps	4.365
Francis 2005 EB (bach)	1.980	de Bethisy	4.510
Neidhardt 3 bps	2.015	Sorge1728	4.549
Sorge 1758 bps	2.015	Neidhardt-3	4.549
Neidhardt-4	2.020	Neidhardt 3 bps	4.582
Sorge1728	2.074	Sorge 1758 bps	4.582
Neidhardt-3	2.074	Neidhardt 4 bps	4.618
Kelletat	2.091	Neidhardt-4	4.692
Sorge 1744 bps	2.140	Jira offen 1 (bach)	4.700

Mobbs/Mackenzie (bach)	2.142	Romieu -1/9 sc	4.907
Billeter	2.190	Meantone -1/9 c	4.907
Sorge1744	2.200	Billeter	4.958
Werckmeister III	2.208	Kelletat	4.965
Werckmeister III bps	2.247	Mobbs/Mackenzie (bach)	5.024
Romieu -1/9 sc	2.355	Romieu -1/10 sc	5.144
Meantone -1/9 c	2.355	"Barthold Fritz?"	5.316
Stanhope bps	2.430	Sparschuh-Zapf (bach)	5.338
Bendeler III bps	2.489	Gothel/Niederbobritzsch	5.397
Sparschuh-Zapf (bach)	2.534	Bendeler-III	5.433
Bendeler-III	2.539	Galileo Galilei 12-TET	5.530
Romieu -1/10 sc	2.548	12TET	5.530
Stanhope	2.633	Fritz: Equal beat rate temperament	5.584
"Barthold Fritz?"	2.684	Werckmeister III	5.672
Bendeler-II	2.693	Bendeler III bps	5.855
Fritz: Equal beat rate temperament	2.699	Werckmeister III bps	5.954
Galileo Galilei 12-TET	2.724	Stanhope bps	5.981
12TET	2.724	Werckmeister II	6.096
Werckmeister II	2.808	Stanhope	6.213
Bendeler-I	3.328	Bendeler-II	6.418

Table 12 : “classification” of Well Temperaments, in function of impurities spread