Bach’s Tempered Meantone (extended version)

Related to “Das wohltemperirte Clavier”

Abstract :

Fifths and major thirds beat rate characteristics of famous historical temperaments are analysed.

It appears that beat rate characteristics might be the actual determining factors for Baroque temperaments, mainly because beat rates are of main importance to interpreting musicians regarding harmony and possible musical affects, and to auditory tuners because of quality and ease of tuning. It is, on the other hand, not always clear whether published ratios, cents or comma’s are deduced from theoretic calculations or from concrete results on monochord measurements or settings.

The revealed reality and importance of beat rate characteristics of temperaments raises additional arguments for acceptability of the Jobin proposal concerning a probable Bach temperament, or for almost identical beat rate alternatives.

A novel hypothesis is proposed concerning the spirals drawn on top of the title page of “Das wohltemperirte Clavier” of Johan Sebastian Bach.

Keywords

Baroque ; well temperament ; meantone ; interval ; comma ; beat ; harmonic ; ratio ; cent ; Bach

1 Preamble

The commonly published and dominating factors with discussions on musical temperaments are probably the investigations on purity deviations of musical intervals, measured in ratios, cents or commas.

And still, musical interval beats and their beating rates are probably more affecting musical factors to interpreting musicians and auditory tuners of keyboard musical instruments.

More attention might therefore have to be paid to those characteristics : beats are undesired and directly observable. Approximate auditory beat rate evaluations do not require any tool nor calculation. Impurity measurements in ratios, cents or commas on the other hand, are often nothing more but rather abstract concepts to many musicians, not of direct use or interest when playing music and also not for auditory tuning.

This paper is an attempt to confirm and elucidate the importance and practical applicability of beat rate evaluations in the determination of musical temperaments, especially some Baroque ones.

2 The auditory music keyboard tuning

The elementary basic concepts of musical temperaments, seen from the point of view of the interpreting musician and the auditory music keyboard tuner are discussed in this paragraph.

There is of course much more that can be written on this subject, see for example : "Le Clavier Bien Obtempéré", A. Calvet, 2020.

People acquainted with the subject of auditory tuning, can skip paragraphs 2.1, 2.2, 2.3.

2.1 The "Reason" at Baroque time

At Baroque time, decimal systems or fractions were not yet currently applied.

Many differing systems existed, often based on duodecimal fractions, for all kinds of measurements : money, length, weight, time,... Still today, this type of measurements is used in some countries, among those not the least developed. Derived measurements, such as volume, surface or speed for example, are even more complex.

The physics of sounds was not known in depth : it was not commonly known or clear yet, that musical sounds are periodic, and consist of a sum of sinusoidal waves.

There was no standard decimal notation of fractions. Some early decimal notation system is described by S. Stevin (1586), and the decimal units or the application of commas or points probably became introduced by G. Rheticus (1542), B. Pitiscus (1613) and J. Napier (1614). The calculation of roots, trigonometric values, logarithms,... was made by hand and very laborious.

The 12TET equal temperament ratios became discussed by Zhu Zaiyu (1536 – 1611) and S. Stevin (1548 – 1620). The latter was probably the first European scientist to calculate the required ratios (ca. 1605). He calculated that string lengths on a monochord should be proportional to the figures displayed in table 1 (no decimals yet !). Verification of the published figures shows some minor corrections are possible.

C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B	c
10000	9438	8908	8409	7936	7491	7071	6674	6298	5944	5611	5296	5000
	9439	8909		7937	7492			6300		5612	5297	

Table 1 : required 12TET string length proportions on a monochord, according S. Stevin (+ minor corrections on second row)

2.2 Pure Musical Intervals (just and perfect intervals)

Music consists of ordained periodic sounds.

Purity of coincident musical sounds is usually desired. Coincident sounds are considered pure, if no beats occur. Beats can occur due to the interference of harmonics of differing periodic sounds.

Any periodic sound can mathematically be simulated by a periodic function $F(t)$.

J. Fourier (1768-1830) developed mathematical evidence that any periodic function $F(t)$ consists of a sum of sine waves, − the harmonics −, whereby ***the sine wave frequencies are INTEGER multiples of a basic frequency***.

$$F(t) = \sum_{n=1}^{\infty} [a_n \sin(2\pi n f t) + b_n \cos(2\pi n f t)] \quad \text{with} \quad n = \mathbb{N}\ ; \quad \text{and with :}$$

$$a_n = 2f \int_{-1/(2f)}^{1/(2f)} F(t) \sin(2\pi n f t)\, dt \quad \text{and} \quad b_n = 2f \int_{-1/(2f)}^{1/(2f)} F(t) \cos(2\pi n f t)\, dt$$

A small musical interval impurity leads to a beating sound, because of ***the summation of mutual note harmonics***, of almost equal frequency. The sum of two sine waves is worked out in the formula below :

$$a\sin(2\pi f_a t) + b\sin(2\pi f_b t) = \sqrt{a^2 + b^2 + 2ab\cos[2\pi(f_a - f_b)t]} \times \cos\left(2\pi \frac{f_a + f_b}{2} t - \psi\right)$$

Hence, this sum corresponds to a single sine wave :

- of median frequency $(f_a + f_b)/2$
- with amplitude modulation from $(a - b)$ to $(a + b)$, at modulation frequency $(f_a - f_b)$
- with (low influence) phase modulation $\psi = \tan^{-1}\left[\frac{a+b}{a-b}\cot\left(2\pi \frac{f_a - f_b}{2} t\right)\right]$

Figure 1 displays the effect.
For example : the beating of an imperfect fifth − ratio ≈ 3/2 −, mainly results from the sum of the second harmonic of the upper note with the third harmonic of the lower note, −but also from any higher harmonic 2n of the upper note with any mutual higher harmonic 3n of the lower note−.
The lowest beat rate of a fifth can therefore be set as :

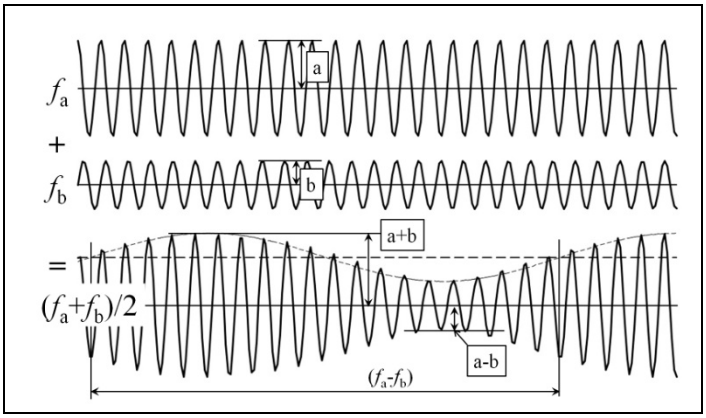

Fig.1 Beating of two sine wave sounds

$$Beat_{\text{fifth}} = 2f_{\text{upper note}} - 3f_{\text{lower note}} = \mathrm{p}_1 f_2 - \mathrm{p}_2 f_1$$

This impurity measurement has been applied already by A. Kellner (1977), and is applicable for other intervals too, applying ***appropriate integer numbers for p_1 and p_2***.

All of the above demonstrates why important musical intervals have ratios of (low) integer numbers.

Besides the absence of beats, there is a reinforcement of the common harmonics, leading to a pleasant and rich new sound : a beautiful "consonance" (= harmony) and also a "resonance".

Observation :
The above is valid only for ideal theoretic harmonic vibrators.

The real world of air column or string vibrators diverges slightly from the ideal theoretic model : the harmonic frequencies differ slightly from an integer multiple of the base frequency due to physical imperfections and boundary conditions, and the harmonic frequencies are usually a little high for string vibrators. The real pitches of auditory tuned temperaments on physical music instruments will therefore differ somewhat from calculated figures. Another consequence, for example, is the octave stretch that can be observed for pianos (Railsback, 1938). A theoretical model of piano strings has been developed by T. Paintoux (Calvet, Annexe 12, pp. 429 – 447). This model is based on the introduction of an inharmonicity factor depending on the string tension, length, diameter, specific weight and elasticity.

2.3 Pythagorean Temperaments

Pythagoras (c. 570 − c. 495 BC) is probably among the firsts to have established that pure musical intervals correspond with simple ratios of integers. Most important musical intervals are the prime with ratio 1/1, the octave with ratio 2/1, the perfect fifth with ratio 3/2, and the perfect fourth, −its inverse−, with ratio 4/3. The just major third has a ratio 5/4, the just minor one has a ratio 6/5.

The perfect fifth with ratio 3/2 leads to what is called Pythagorean tuning.

This tuning in turn, leads to pentatonic and heptatonic/diatonic musical scales, and further on also chromatic scales. A possible pentatonic scale can, for example, be obtained by consecutive fifths on notes F, C, G, D, leading to the A note, and building the C – D – F – G – A scale. A possible heptatonic scale can be obtained by adding fifths on A and E leading to the B note, building the C – D – E – F – G – A – B scale. Further extension to chromatic scales can be achieved by additional ascending fifths, leading to notes F#, C#, G#, etc…, and additional descending fifths, leading to notes B*b*, E*b*, A*b*, etc… A possible chromatic scale can be, for example,
C – C# – D – E*b* – E – F – F# – G – G# – A – B*b* – B.

It can be observed that from the twelfth note on, some notes obtain very similar pitches, such as for notes G# and A*b*, for example. Those notes are called enharmonic, and differ by what is called a Pythagorean comma, ratio $3^{12}/2^{19}$ (= 1.013643265…).

The Pythagorean comma leads to a disturbing beat rate on the fifth “closing” the circle of fifths. This circle of fifths becomes often closed by the G#-D# fifth (on a “enharmonic keyboard”, this is also : G#-E*b*, or A*b*-E*b*) with a disturbing beat rate of - 16.86... beats/sec (for A = 440), instead of the zero beats/sec. of the perfect fifths.

2.4 Equal Temperament

The Pythagorean temperament was probably the dominating temperament applied in practice, until the begin of the 16-th century. Alternative temperaments intervened, to avoid a twelfth “closing” fifth with disturbing beat rate. Among those the so called equal temperament. Vincenzo Galilei (c. 1520 – 1591) was probably among the firsts to apply an equal temperament.

Doubts can be expressed on which equal temperament was installed. The 12TET is commonly accepted nowadays. But the required ratios are not simple to set precisely, and the only available instrument at Baroque time to assist a tuner to set this temperament was the monochord.

Even today, this is not easy : a deviation of only 1 mm. on a string of 1 m. corresponds to a deviation of $1200 \times \log_2(1001/1000) = \sim 2$ cents, and on top of that one has to count with the inharmonicity of any real physical string or organ pipe. Precise electronic measurements show that measured string pitches might be unstable or vary in time. A fluctuation up to 0.22 Hz on the F3 note (173.87 to 174.09 Hz, this is ∼ 2 cent !), was observed by Calvet (2020, p. 282 – 284). He reports even up to 8 cents, due to a glitch in the measured values, but this might probably be due to minor deficiencies of the software measuring algorithms (probably FFT).

The German term for equal temperament is “Gleichschwebende Temperatur” (equally beating temperament). Therefore, one can think indeed of a temperament whereby all fifths have an equal beat rate instead of the equal and slightly reduced fifths ratio of the 12TET. An equal beat rate temperament was probably proposed by B. Fritz (Fritz 1756 ; Kroesbergen 2013, p. 16 – 20).

Auditory keyboard tuning is often initialised setting the notes on a scale from F3 to F4 (Calvet, 2020). The notes within this scale have rather low pitches and contain many harmonics, facilitating the tuning because of clear and low beating rates of intervals. Moreover, the major third on C will have the best ratio of all thirds, if all thirds have equal beating rate, because this major third has the highest pitch due to its position of the F3 – F4 scale. The fifths within this equal beat rate scale can be calculated based on the equations table 2, whereby q_{Note} stands for the beat rate of the fifth on that note. An equal beat rate on all fifths, this is by setting all the $q_{Note} = Beat$, leads to a linear set of 12 equations containing 12 variables. Hence, calculation of the notes and the beat rate is easy.

$3F3 - 2C4 + q_F = 0$	$3C4 - 4G3 + q_C = 0$	$3G3 - 2D4 + q_G = 0$	$3D4 + q_D = 4A3$
$-2E4 + q_A = -3A3$	$3E4 - 4B3 + q_E = 0$	$3B3 - 4F\#3 + q_B = 0$	$3F\#3 - 2C\#4 + q_{F\#} = 0$
$3C\#4 - 4G\#3 + q_{C\#} = 0$	$3G\#3 - 2Eb4 + q_{G\#} = 0$	$3Eb4 - 4Bb3 + q_{Eb} = 0$	$3Bb3 - 4F3 + q_{Bb} = 0$

Table 2 : calculation of fifths beating rate within the F3 – F4 scale

A Baroque diapason of A4 = 415 (A3 = 207.5), leads, within the F3 − F4 scale, to a fifths beating rate of $Beat$ = − 0.75… beats/sec. The obtained interval beat rates are displayed in table 3.

		C4	C#4	D4	E*b*4	E4	F4	F#4	G4	G#4	A4	B*b*4	B4
12TET	Pitches	246.76	261.43	276.98	293.45	310.90	329.39	348.97	369.72	391.71	415.00	439.68	465.82
	fifth beats	– 0.84	– 0.89	– 0.94	– 0.99	– 1.05	– 1.12	– 1.18	– 1.25	– 1.33	– 1.41	– 1.49	– 1.58
Fritz	Pitches	246.73	261.43	276.92	293.45	310.87	329.48	349.08	369.72	391.77	415.00	439.80	465.94
	fifth beats	– 0.75	– 0.75	– 0.75	– 0.75	– 0.75	– 1.50	– 1.50	– 1.50	– 1.50	– 1.50	– 1.50	– 1.50
	Δ pitches	– 0.03	0.00	– 0.06	0.00	– 0.02	0.09	0.10	0.00	0.06	0.00	0.13	0.11

Table 3 : Comparison of the 12TET and Fritz equal temperaments (for A4 = 415) ;
notice the jump with a factor 2 for the beat rate, on F4

It is clear that up to twelve differing equal beating temperaments can be obtained, depending on the chosen initial note for the twelve step tuning scale, because a differing initial note goes at par with differing equations. ***The equal beating rate also alters with a factor two, for every octave step***, regardless the chosen initial note or diapason (see for example : note F4, table 3).

Auditory tuning instructions for an equal beat rate temperament are very simple : let all fifths within F3 to F4 beat at the prescribed equal beat rate. However, a very high tuning precision is required, in order to obtain an acceptable closure of the circle of fifths.

Corresponding auditory tuning instructions for the 12TET are more elaborate because of twelve differing beat rates, or, the tuning requires the use of a monochord or pitch measuring instruments, what goes at par with all already mentioned difficulties.

The above considerations, discussing auditory tuning based on equal beat rates, could put in doubt that early installed equal temperaments correspond with the 12TET. But it must be admitted : both equal temperaments are very comparable. The auditory differences are indistinguishable. The 12 TET mean beat rate (within the F3 − F4 scale), is − 0.78... beats/sec. ; this is only slightly more and almost identical to the − 0.75… beats/sec of the Fritz temperament.

2.5 The Meantone

Just major thirds, ratio 5/4, are in general appreciated and desired by musicians and their audience. The quality of major thirds can determine the differing characters or affects of differing temperaments. Hence, much importance is paid to the just major third. Just major thirds differ from the rather sharp ones of the Pythagorean tuning, having a ratio 81/64. The difference is the syntonic comma, with ratio 81/80. Early considerations on the importance of the just major thirds, can, among others, be attributed to Ptolemaeus (c. 100 − c. 170).

The just major thirds became widely introduced and accepted because of the meantone temperament. This temperament was described in 1523 by P. Aaron and by Salinas (1577) according Zarlino. This meantone is based on a just major third on C that is divided in four equal halftones. This temperament is therefore called the "quarter (syntonic) comma meantone". All other notes emanate from just major thirds on the notes E*b*, B*b*, F, (C), G, D, A, E. This meantone was probably the "dominating" temperament for Baroque music (approximately 1600 to 1750).

For the meantone also, one may wonder whether the C − E major third should be divided based on ratios, cents or commas, as is commonly assumed, presumably by means of a monochord or measuring instrument ?

A hypothetical alternative division might consist of building the just major third C − E, based on four fifths with equal beat rate (E − A, D − A, G − D, C − G), followed by the further installation of the seven additional and desired just major thirds. This determination of notes also has the advantage it also contains a direct link with the A note when initiating the tuning, but also the C note, and this imports if the C note is chosen to set the diapason. In analogy with table 2, a set of equations applicable on major thirds beat rates can be set up. See table 4.

$5F3-4A3+p_F=0$	$5C4-4E4+p_C=0$	$5G3-4B3+p_G=0$	$5D4-8F\#3+p_D=0$
$-4C\#4+p_A=-5A3$	$5E4-8G\#3+p_E=0$	$5B3-4Eb4+p_B=0$	$5F\#3-4Bb3+p_{F\#}=0$
$5C\#4-8F3+p_{C\#}=0$	$5G\#3-2C4+p_{G\#}=0$	$5Eb4-8G3+p_{Eb}=0$	$5Bb3-4D4+p_{Bb}=0$

Table 4 : calculation of major thirds beating rate within the scale F3-F4

Calculation of the meantone note pitches : see table 5, with equations selected from tables 2 and 4.

$3C4-4G3+Beat=0$ $3G3-2D4+Beat=0$	$3D4+Beat=4A3$ $-2E4+Beat=-3A3$	$5C4-4E4=0$
$5F3-4A3=0$ $5Bb3-4D4=0$ $5Eb4-8G3=0$	$5G3-4B3=0$ $5D4-8F\#3=0$	$-4C\#4=-5A3$ $5E4-8G\#3=0$

Table 5 : Equations leading to a meantone temperament with four equally beating fifths, and eight just thirds :
Upper row : building of the just major third on C Lower row : expansion with just major thirds

The beat rate of the initial four fifths amounts to − 2.09... beats/sec. Obtained pitches : see table 6

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
1/4 comma Meantone	Pitches	248.23	259.38	277.53	296.95	310.28	332.00	346.91	371.19	387.86	415.00	444.04	463.98
"Equal beat Meantone"	Pitches	248.17	259.38	277.36	296.96	310.21	332.00	346.70	371.21	387.76	415.00	443.78	464.01
	Δ-pitches	− 0.06	0.00	− 0.17	0.01	− 0.08	0.00	− 0.21	0.02	− 0.10	0.00	− 0.26	0.02
	Δ-cents	0.43	0.00	1.03	− 0.09	0.43	0.00	1.03	− 0.09	0.43	0.00	1.03	− 0.09

Table 6 : comparison between the "classic" 1/4 comma meantone, and the "equal beat" meantone

This proposed alternative auditory tuning could be installed at much ease by any Baroque tuner, not using any measuring tool at all. Therefore doubts can be expressed on the effective installation during the Baroque period, of the nowadays commonly published pitches of the meantone temperament, based on ratio calculations.

Precise tuning of the "classic" division of the just C−E third was only possible by means of a monochord, but this goes at par with quite intensive labour : for a monochord of length of 1000 mm., tuned on the note C, the movable bridge must be set at C# = 935.73 mm, D = 893.33 mm, E*b* = 835.90 mm, and E = 800.00 mm. A deviation of only 1 mm corresponds to a pitch deviation of 2.16… cents already. This is more than twice the maximum differences displayed at table 6. Hence, there is in fact no single practical reason not to tune by the ear. The differences between both versions are practically and auditory not distinguishable.

2.6 Well Temperaments

There has been a general and very active quest on well temperaments during the Baroque period. Reason for this is that the meantone does not allow for acceptable musical modulation in all keys, because of the "wolf fifth" (usually on G#) and the associated four "harsh" major thirds.

Please notice :

ALL allowed meantone keys* (Bb, F, C, G, D, A,) *offer EXCELLENT and IDENTICAL HARMONY, the only difference between the allowed keys consists of a difference in pitches only.
Therefore, modulation is musically completely free, as long as it remains within the allowed keys.

Werckmeister was, among others, at the origin of the well temperament concept, and his well tempered Werckmeister III (1691) temperament became famous. There are however precedents, for example : the reconstruction of the organ of the Cathedral of Lucca, Italy 1473 (Devie 1990, p. 55) testifies to the requirement of a well temperament.

Werckmeister is probably also the first to have used the "wohltemperiert" (well tempered) term in writing. See the figures 2 (Norback, 2002, p 18), and 3 to 7.

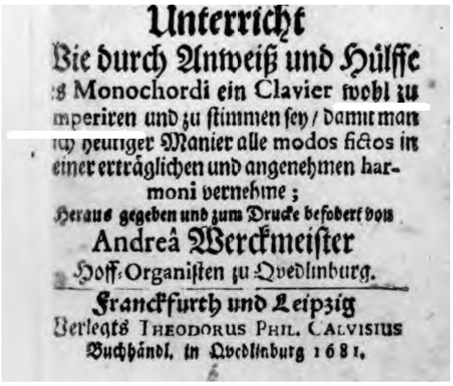
Unterricht
Wie durch Anweiß und Hülffe
s Monochordi ein Clavier wohl zu
mperiren und zu stimmen sey / damit man
heutiger Manier alle modos fictos in
einer erträglichen und angenehmen har-
moni vernehme ;
Heraus gegeben und zum Drucke befodert von
Andreâ Werckmeister
Hoff-Organisten zu Quedlinburg.
Franckfurth und Leipzig
Verlegts Theodorus Phil. Calvisius
Buchhändl. in Quedlinburg 1681.

Fig. 2 : Orgelprobe 1681, title page

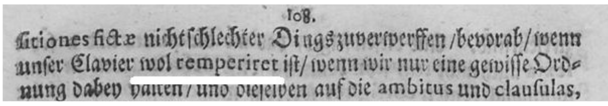
108.
fictiones fictæ nicht schlechter Dings zuverwerffen / bevorab / wenn
unser Clavier wol temperiret ist / wenn wir nur eine gewisse Ord-
nung dabey halten / und dieselben auf die ambitus und clausulas,

Fig. 3 : Musicae Hodegus Curiosus , 1686, chapter 30 page 118 {erratically marked as 108)

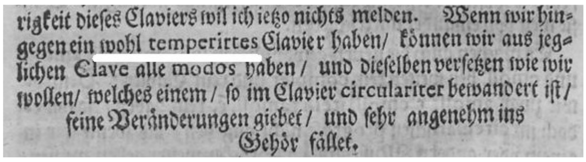
rigkeit dieses Claviers wil ich ietzo nichts melden. Wenn wir hin-
gegen ein wohl temperirtes Clavier haben / können wir aus jeg-
lichen Clave alle modos haben / und dieselben versetzen wie wir
wollen / welches einem / so im Clavier circulariter bewandert ist /
feine Veränderungen giebet / und sehr angenehm ins
Gehör fället.

Fig. 4 : Musicae Hodegus Curiosus , 1686, p. 120, 16-th rule


Musicalische Temperatur,
Oder
deutlicher und warer Mathematischer Unterricht /
Wie man durch Anweisung des
MONOCHORDI
Ein Clavier / sonderlich die Orgel-Wercke /
Positive, Regale, Spinetten / und dergleichen wol tempe-
rirt stimmen könne / damit nach heutiger manier alle Modi ficti
in einer angenehm- und erträglichen Harmonia mögen
genommen werden /

Fig. 5 : Musicalische Temperatur, 1691, title page

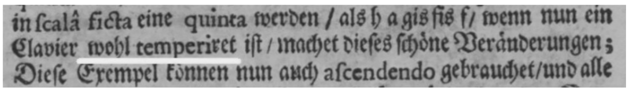
in scalâ ficta eine quinta werden / als h a gis fis f / wenn nun ein
Clavier wohl temperiret ist / machet dieses schöne Veränderungen ;
Diese Exempel können nun auch ascendendo gebrauchet / und alle

Fig. 6 : Musicalische Temperatur, 1691, chapter 22, page 61, 7-th rule

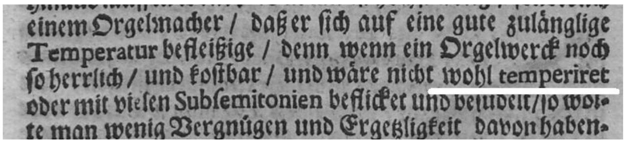
einem Orgelmacher / daß er sich auf eine gute zulängliche
Temperatur befleißige / denn wenn ein Orgelwerck noch
so herrlich / und kostbar / und wäre nicht wohl temperiret
oder mit vielen Subsemitonien beflicket und besudelt / so wor-
te man wenig Vergnügen und Ergetzligkeit davon haben.

Fig. 7 : Orgelprobe 1698, p. 7

A musical definition of well temperaments, based on the Werckmeister criteria, was elaborated by H. Kelletat[1] (1960 ; 1981, p. 9) :

> << Well temperament means a mathematical-acoustic and musical-practical organisation of the tone system within the twelve steps of an octave, so that impeccable performance in all tonalities is enabled, based on the extended just intonation (natural-harmonic tone system), while striving to keep the diatonic intervals as pure as possible.
> This temperament acts, while tied to given pitch ratios, as a thriftily tempered smoothing and extension of the meantone, as unequally beating half tones and as equal (equally beating)

[1] "Wohltemperierung heißt mathematisch-akustische und praktisch-musikalischen Einrichtung von Tonmaterial innerhalb der zwölfstufigen Oktavskala zum einwandfreien Gebrauch in allen Tonarten auf der Grundlage des natürlich-harmonischen Systems mit Bestreben möglichster Reinerhaltung der diatonische Intervalle.
Sie tritt auf als proportionsgebundene, sparsam temperierende Lockerung und Dehnung des mitteltönigen Systems, als ungleichschwebende Semitonik und als gleichschwebende Temperatur."

temperament. >>

Nowadays again, well temperaments (= circulating temperaments) have become a hot musical topic.

It is quite probable that the publications of H. Kelletat (1956, 1960, 1981, 1982,) are at the origin of the present interest. A more recent publication also, of A. Calvet (2020), offers considerations in width and in depth on the musical temperament and tuning topic, supported by historical aspects and profound explanation on how and why musical temperaments, intervals and interval beat rates have specific important characteristics. Calvet has treated in particular and depth also the aspect of musical interval beating, and the beat rates of many temperaments are well documented. He is also discussing the importance of required interval readjustments during the practice of piano tuning, because of inharmonicity of strings, or corrections for better distribution of beats (and these corrections might therefore be due to desired auditive beat rate improvements leading to deviations from published pitches). Jobin (2005) also, mentions the necessity of interval readjustments.

2.7 **Werckmeister** (1635 − 1706)

Werckmeister has published his tuning instructions, based on commas, see table 7 concerning Werckmeister III (1698, chap. 30, p. 78), his most applied and famous temperament. This temperament also, can be recalculated, based on beat rates, by means of the equations table 2, setting $q_C = q_D = q_G = q_B = Beat$, and all other $q_{Note} = 0$. The beat rate of the fifths on notes C4, G3, D4, B3 is − 2.35... beats/sec. The differences between the published and recalculated versions are minimal, and can very probably not be distinguished auditory (see table 7).

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Werckm. III	Pitches	248.44	261.73	277.61	294.45	311.25	331.25	348.97	371.40	392.59	415.00	441.67	466.88
	commas	− 1/4	0	− 1/4	0	0	0	0	− 1/4	0	0	0	− 1/4
beat rate Werckm.	Pitches	248.45	261.74	277.45	294.45	311.25	331.26	348.98	371.50	392.61	415.00	441.68	466.88
	Δ-pitches	0.01	0.01	− 0.16	0.01	0.00	0.01	0.01	0.10	0.01	0.00	0.01	0.00

Table 7 : comparison between the "classic" Werckmeister III, and the "equal beat" Werckmeister III

2.8 **Vallotti** (1697 − 1780)

The Vallotti temperament is characterised by equality of diminished diatonic fifths, all other fifths being perfect. It is part of the countless amount of well temperaments created at Baroque time. The diminished fifths lead to some rather improved but still not yet just diatonic major thirds.

For Vallotti also, a comparison is possible between the temperament based on equality of fifths in cents, and the one based on equality in beat rates. The "beat rate Vallotti" is easy to calculate by means of the equations table 2, with $q_F = q_C = q_G = q_D = q_A = q_E = Beat$, and all other $q_{Note} = 0$.

For Vallotti too, the tuning procedure based on equal fifths beat rates is much simpler than the one based on ratios. The applicable fifths beat rate is − 1.50... (within F3 − F4)

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Vallotti	Pitches	247.60	261.43	277.29	294.11	310.55	330.88	348.58	370.56	392.15	415.00	441.17	464.77
Beat rate Vallotti	Pitches	247.53	261.56	277.17	294.26	310.50	331.04	348.75	370.55	392.35	415.00	441.39	465.00
	Δ-pitches	−0.06	0.13	−0.13	0.15	−0.05	0.17	0.17	−0.01	0.20	0.00	0.22	0.23

Table 8 : comparison between the "classic" Vallotti, and an "equal beat" Vallotti

Both Vallotti scales are almost identical. The differences between both versions are auditory indistinguishable. For Vallotti, too, it is much simpler to tune for equal harmonic beat rates on fifths than to tune for equal ratios. The applicable fifths beat rate is - 1.50 ... (within the F3 – F4 limits).

2.9 Kirnberger III

Kirnberger III is a commonly known well temperament, and is probably one of the most famous well temperaments

Kirnberger temperaments are characterized by a pure third on the C note, obtained thanks to an adequate temperament of relevant fifths. All the other fifths thereafter are pure, except the one on F #, which should not be tuned, but which thus obtains a weak residual beat rate.

The historical Kirnberger I version (1766) holds a single diminished fifth on the D note, which leads to four pure major thirds ; those on F, C, G, D. This version was rejected because of an excessive reduction of the fifth on D. The historical Kirnberger II version (1771) holds a shared diminishing of fifths on the D and A notes, which leads to three pure major thirds still, those on C, G, D, and which was also strongly criticized for its diminished fifths, still considered excessive.

The Kirnberger III is characterized by a just major C − E third, and all fifths perfect, except those involved in building the C − E third and the one on F#. Two versions exist : Kirnberger III, and Kirnberger III unequal (= ungleich), (Kelletat, 1981, p. 158, table 12). The "common" Kirnberger III version has ***equal impurity, expressed in ratios or cents***, for the fifths on C, G, D, A building the just C − E major third. The Kirnberger III ungleich version has ***unequal ratios for those fifths***, but, surprisingly, those have almost equal beat rates, except for the fifth on C.

It is the above observation that lead to the recalculation of Kirnberger III ungleich, based on beat rates. This in turn lead to the alternative calculation of the Jobin Bach (see further), the present paper, and the hypotheses about beat rates and their importance in the development of historical temperaments.

See table 9 displaying the applicable formulas taken from tables 2 and 4, to recalculate Kirnberger III ungleich. It should be noticed this calculation of notes C4, G3, D4, A3, E4 is identical to their calculation for the "beat rate" meantone (see table 6) : − 2.09... beats/sec for the involved fifths. Further equations correspond with the desired perfect fifths.

$3C4 - 4G3 + Beat = 0$ $3G3 - 2D4 + Beat = 0$	$3D4 + Beat = 4A3$ $-2E4 + Beat = -3A3$	$5C4 - 4E4 = 0$
$3E4 - 4B3 = 0$ $3B3 - 4F\#3 = 0$	$3F3 - 2C4 = 0$ $3Bb3 - 4F3 = 0$ $3Eb4 - 4Bb3 = 0$	$3G\#3 - 2Eb4 - 0$ $3C\#4 - 4G\#3 = 0$

Table 9 : Equations leading to four equally beating fifths, and consecutive seven perfect fifths :

The proposed temperament is compared with the commonly published Kirnberger III versions, in table 10. The differences between the versions are probably auditory indistinguishable.

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Kirnberger III	Pitches 1	248.23	261.51	277.53	294.20	310.28	330.97	349.07	371.19	392.26	415.00	441.29	465.43
Kirnb. III ungl	Pitches 2	248.15	261.43	277.77	294.11	310.19	330.87	348.96	371.43	392.14	415.00	441.16	465.28
Beat rate Kirnberger III	Pitches	248.17	261.44	277.36	294.12	310.21	330.89	348.98	371.21	392.16	415.00	441.18	465.31
	Δ-pitch 1	− 0.06	− 0.07	− 0.17	− 0.07	− 0.08	− 0.08	− 0.09	0.02	− 0.10	0.00	− 0.11	− 0.12
	Δ-pitch 2	0.01	0.02	− 0.41	0.02	0.02	0.02	0.02	− 0.22	0.02	0.00	0.03	0.03

Table 10 : comparison between the "classic" Kirnberger III, Kirnberger III ungleich, and the recalculated one

The consecutive Kirnberger temperament characteristics, may allow to assume those are the fruit of a desire to follow Bach's recommendations (cf. Kelletat and many other sources), by maintaining a pure major C – E third, followed by fast and easy tuning of fifths. The Kirnberger III ungleich temperament is probably the one that is most easy to set in case of auditory well tempered tuning : as soon as the just major third C − E is set, it is sufficient to set seven more perfect fifths, the longest chain of fifths containing only five of those (those on F, B*b*, E*b*, G#, and C#).

2.10 Other historic temperaments

More historic temperaments can be analysed, if reliable, adequate and documented historic information is available, the early Baroque ones being the most interesting. Most consulted data originate from De Bie (2001, most temperaments are based on Barbour). See Appendix A for details on a number of recalculations.

Possible observations on the results of Appendix A :

- All obtained RMS−Δ−cent display evidence of close fits between pitches of the "classic" temperament and the beat rate calculated ones. Only three show a very slightly lesser fit.
- Stanhope is the only one with a just major third on C ***and*** G, alike Jobin (see table 14).

Besides all the so far mentioned temperaments, there are a multitude of other temperaments created during the Baroque period. Probably often part of a well temperament quest.

3 Bach

A quest is still going on, about what might have been the temperament applied by Bach, when tuning a keyboard. It might have been any temperament, and the possibility for ***some well temperament*** arises for sure following his visit to Buxtehude (1705). A possible and general recognised outcome for this quest of what might have been "his" well temperament, might therefore be of historical and musicological relevance.

Discussions on Bach temperaments can be controversial, due to the fact that J. S. Bach has not left any written instructions on how to tune a keyboard. No historic certainty at all exists on how he tuned his clavichord, although it is mentioned and generally accepted he was very skilled at it, and extremely rapid.

He was for sure very sensitive to musical affects : he enlightened the qualities of well temperaments by means of "Das wohltemperirte Clavier" (1722), but he also expressed horror in some parts of the "St. Matthew Passion" (1727), by intentional application of "forbidden" meantone keys (E, B, F#, C#, G#, E*b*) on meantone tuned instruments (Kelletat 1982, p. 20).

Observation : the "St. Matthew Passion" (1727), where meantone can have a major impact on some musical affects, is POSTERIOR to "Das wohltemperirte Clavier" (1722) based on well tempering.

Doubts on the application of the 12 TET by Bach were expressed by Kirnberger already (Kelletat, 1981, p. 40–42, letters of Kirnberger to Forkel 1779–1780, rejecting Marpurg's opinion ; Marpurg, 1776). Nevertheless, Marpurg's opinion on the application of the 12TET by Bach gained general acceptance and was copied over a long period of time, almost two centuries, in a countless amount of publications.

Doubts on the 12TET application by Bach intervined again in modern times, probably first with Bosanquet (1876, p. 28–30), and later on Kelletat (1957, 1960, 1981, 1982), the latter probably being the lead to a breakthrough on those doubts nowadays. Kelletat believes the temperament that was applied by Bach could have been Kirnberger III, ***OR ANY OTHER SIMILAR ONE***. The opinion ***“ or any other similar one “*** must probably be preferred, because of a number of dates. Indeed, “Das wohltemperirte Clavier “ dates from 1722, Kirnberger was his pupil from 1739 until 1741, Bach decesed in 1750, the Kirnberger I temperament dates from 1761, and Kirnberger II from 1771. Many possible Bach tuning schemes have been published since Kelletat’s publication, probably starting with Kellner (1977).

Sparschuh (1999) was probably the first to publish a hypotheses that Bach left a tuning instruction message in spirals drawn on top of a score of “Das wohltemperirte Clavier” ; see fig. 8 (Amiot, 2008 touched up copy : addition of note names on top of the spirals). Alternative interpretations were proposed later on (Zapf 2001, Lehman 2005, Jobin 2005, and many more), in which Lehman became widely observed and discussed.

Jobin (2005) attributed specific impurity qualities to fifths in function of the type of spirals, based on a normal, but inverse, musical sequence of the fifths, such as displayed in fig. 8. The fifths on C, G, D, A, E, should have ***equal impurity***, so to ***build a just major third on C*** ; this impurity must therefore amount to one quarter of a syntonic comma. The fifths on G#, E*b*, B*b* should also have ***equal impurity***, so that all remaining fifths should be perfect. A just major third is obtained on the G note also, if fifths impurities are calculated in ratios, as was done by Jobin. Sparschuh and Zapf made similar assumptions, but presuming ***predefined beat rate impurity*** values, what leads to uncertainties concerning obtained results.

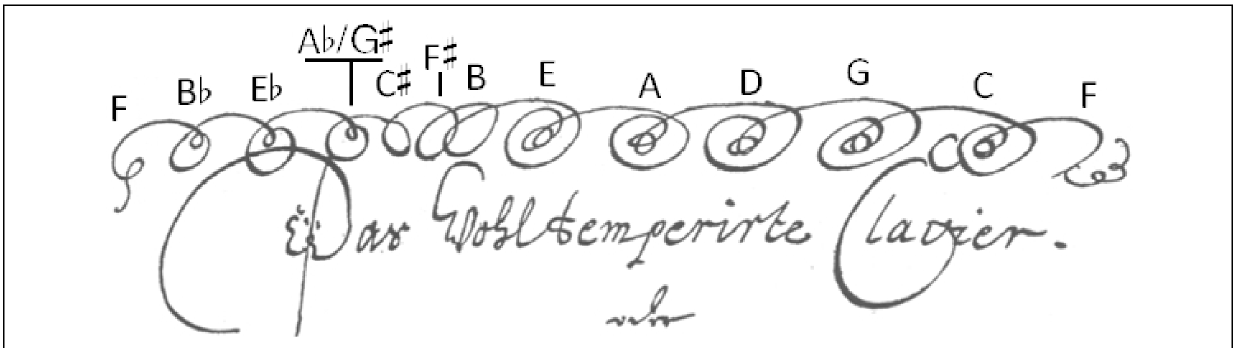

Fig. 8 : Scrolls on the score of “Das wohltemperirte Clavier”

3.1 Calculation of a “Bach-temperament”, based on interval beat rate evaluations

For the Jobin−Bach also, it is easy to calculate a beat rate alternative, whereby the equal impurities do not mean equality in deviation in cents, ratios or commas, but equality in beat rate : see table 11, containing the selected applicable equations of tables 2 and 4.

$3C4 - 4G3 + Beat1 = 0$ $3G3 - 2D4 + Beat1 = 0$	$-2E4 + Beat1 = -3A3$ $3D4 + Beat1 = 4A3$	$3E4 - 4B3 + Beat1 = 0$ $5C4 - 4E4 = 0$	
$3F3 - 2C4 = 0$	$3B3 - 4F\#3 = 0$	$3F\#3 - 2C\#4 = 0$	$3C\#4 - 4G\#3 = 0$
$3G\#3 - 2Eb4 + Beat2 = 0$	$3Eb4 - 4Bb3 + Beat2 = 0$	$3Bb3 - 4F3 + Beat2 = 0$	

Table 11 : Calculation of Bach−C
First row : requirements on fifths to build a just major third on C
Second row : requirements on perfect fifths
Third row : requirements on remaining fifths

The obtained solution is :

$$\frac{-Beat1}{4} = \frac{C4}{476} = \frac{D4}{532} = \frac{E4}{595} = \frac{G3}{356} = \frac{A3}{383}$$

The obtained beat rates are : *Beat1* = − 2.09, *Beat*2 = 0.26

The obtained scale, called Bach−C here, because of the just third on C, is compared with Jobin in table 14. To be in line with former Baroque temperament calculations, the comparison is worked out for a diapason of A = 415. Comparison of scales, demonstrates very good similarity with the Jobin proposal.

3.2 Bach, with just major third on C and G

A more profound analysis is required : ***of ALL the already published "Bach temperament" proposals, NONE indeed, has yet gained general acceptance***. Jobin gives strong musical justification for his hypotheses, based on many arguments, among those also the relation with the meantone, a temperament that was accepted and familiar to Bach ; see above, and see also Kelletat (1981, p. 21, lines 3 to 9) : the just major third on C has identical division in four halve tones for the mean tone, Jobin and Kirnberger III.

The Jobin hypotheses also leads to a just third on G, because of calculations with ratios. And therefore, beat rate alternatives including a just major third on C and G might be of interest. Two alternatives with a just major third on C and G are elaborated : one with a slightly deviating fifth beat rate on C (Bach−dC) and one with a slightly deviating fifth beat rate on E (Bach−dE) ; see tables 12, 13.

$5G3 - 4B3 = 0$ $3G3 - 2D4 + Beat1 = 0$	$-2E4 + Beat1 = -3A3$ $3D4 + Beat1 = 4A3$	$3E4 - 4B3 + Beat1 = 0$ $5C4 - 4E4 = 0$

Table 12 : Bach−dC : requirements on interval for a just major third on C and G, and deviating fifth on C
additional conditions : see row 2 and 3 of table 11

$3C4 - 4G3 + Beat1 = 0$ $5G3 - 4B3 = 0$	$-2E4 + Beat1 = -3A3$ $3D4 + Beat1 = 4A3$	$3E4 - 4B3 + Beat1 = 0$ $5C4 - 4E4 = 0$

Table 13 : Bach−dE : First row : requirements on interval for a just major third on C and G, and deviating fifth on G
additional conditions : see row 2 and 3 of table 11

The equations displayed in tables 12 and 13 must be supplemented with those on rows 2 and 3 of table 11.

The obtained solutions are :

$$\text{Bach-dC :} \quad \frac{-Beat1}{5} = \frac{D4}{635} = \frac{E4}{710} = \frac{G3}{425} = \frac{A3}{475}$$

With following beat rates : *Beat1* = − 2.18 *Beat2* = 0.28 *deviating beat on C4* = − 1.75

$$\text{Bach-dE :} \quad \frac{-Beat1}{4} = \frac{C4}{476} = \frac{D4}{532} = \frac{G3}{356} = \frac{A3}{383}$$

With following beat rates : *Beat1* = − 2.09 *Beat2* = 0.39 *deviating beat on E* = − 2.61

Table 14 displays the obtained pitches.

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Jobin Bach	Pitches	248.23	260.99	277.53	293.81	310.28	330.97	347.99	371.19	391.49	415.00	441.00	463.98
Bach-C	Pitches	248.17	261.15	277.36	293.92	310.21	330.89	348.20	371.21	391.73	415.00	441.01	464.27
	Δ-pitches	− 0.06	0.16	− 0.17	0.11	− 0.08	− 0.08	0.21	0.02	0.24	0.00	0.01	0.28
Bach-dC	Pitches	248.13	261.08	277.39	293.86	310.16	330.84	348.11	371.32	391.62	415.00	440.93	464.14
	Δ-pitches	− 0.10	0.09	− 0.13	0.05	− 0.13	− 0.14	0.12	0.13	0.14	0.00	− 0.08	0.16
Bach-dE	Pitches	248.17	261.00	277.36	293.82	310.21	330.89	348.01	371.21	391.51	415.00	440.93	464.01
	Δ-pitches	− 0.06	0.01	− 0.17	0.01	− 0.08	− 0.08	0.02	0.02	0.02	0.00	− 0.08	0.02

Table 14 : comparison between the "Jobin" Bach, and calculated "beat rate" ones

All C-major diatonic notes of the Bach–dE, except the F note, have pitches identical to the corresponding meantone notes.

Table 15 displays the beat rates of the obtained Bach alternatives, according the sequence of fifths on the spirals of fig. 8.

	F3	B*b*3	E*b*3	A*b*3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Jobin Bach	2,57	7.60	15,71	14,20	18,93	12,04	15,27	14,52	6,46	4,32	0,00	0,00	5,15
	0.00	0.44	0.58	0.39	0.00	0.00	0.00	– 2.89	– 1.93	– 2.58	– 1.73	– 2.31	0.00
	– 14.20	– 18.05	– 22.91	– 14.50	– 14.52	– 6.46	– 4.32	– 5.77	– 3.86	– 10.31	– 11.05	– 20.32	– 28.39
Bach–C	2.78	6.92	15.21	13.35	17.80	11.52	15.02	15.87	7.10	6.00	0.52	0.00	5.56
	0.00	0.26	0.26	0.26	0.00	0.00	0.00	– 2.09	– 2.09	– 2.09	– 2.09	– 2.09	0.00
	– 13.35	– 17.28	– 22.53	– 14.51	– 15.87	– 7.10	– 6.00	– 5.21	– 4.17	– 9.73	– 11.09	– 19.38	– 26.70
Bach–dC	2.91	7.26	15.98	13.45	17.93	11.58	15.07	15.70	6.83	5.46	0.00	0.00	5.82
	0.00	0.28	0.28	0.28	0.00	0.00	0.00	– 2.18	– 2.18	– 2.18	– 2.18	– 1.75	0.00
	– 13.45	– 17.37	– 22.60	– 14.50	– 15.70	– 6.83	– 5.46	– 4.37	– 4.37	– 10.19	– 11.63	– 19.47	– 26.90
Bach–dE	2.78	7.13	15.71	13.90	18.53	11.84	15.27	14.99	6.52	5.21	0.00	0.00	5.56
	0.00	0.39	0.39	0.39	0.00	0.00	0.00	– 2.61	– 2.09	– 2.09	– 2.09	– 2.09	0.00
	– 13.90	– 17.76	– 22.91	– 14.50	– 14.99	– 6.52	– 5.21	– 5.21	– 4.17	– 9.73	– 11.30	– 19.88	– 27.79

Table 15 : beat rates ; up–down by temperament : major thirds, fifths, minor thirds

In between those two, many others can of course be thought of (see for instance the appendix B–B1).

A temperament with mathematical optimisation for best beat rate equality of the fifths on C, G, D, A, E is worked out in appendix B–B1 of this paper. The differences with the table 14 and 15 are minimal (see also the “Cervoreille {≈ brain-ear)” chapter of Calvet, 2020).

A version with best possible equilibrium between major thirds and fifths impurities is worked out in appendix B–B2. The same is done, based on rations calculations in appendix C–C1

A graphical presentation of beat rate properties of major thirds and fifths of the Jobin Bach and the beat rate Bach scales is displayed in fig. 9, with also the meantone fifths beat rates and the Kirnberger III intervals.

It is clear that the major thirds are the determining factor for the characterisation of temperaments. Very small changes in fifths characteristics induce remarkable changes for major thirds characteristics. The strict mathematical beat rate equality of fifths is therefore probably not what imports ; what could import really, is very probably the setting of just major thirds on C and G. Moreover : the purity of both major thirds on C

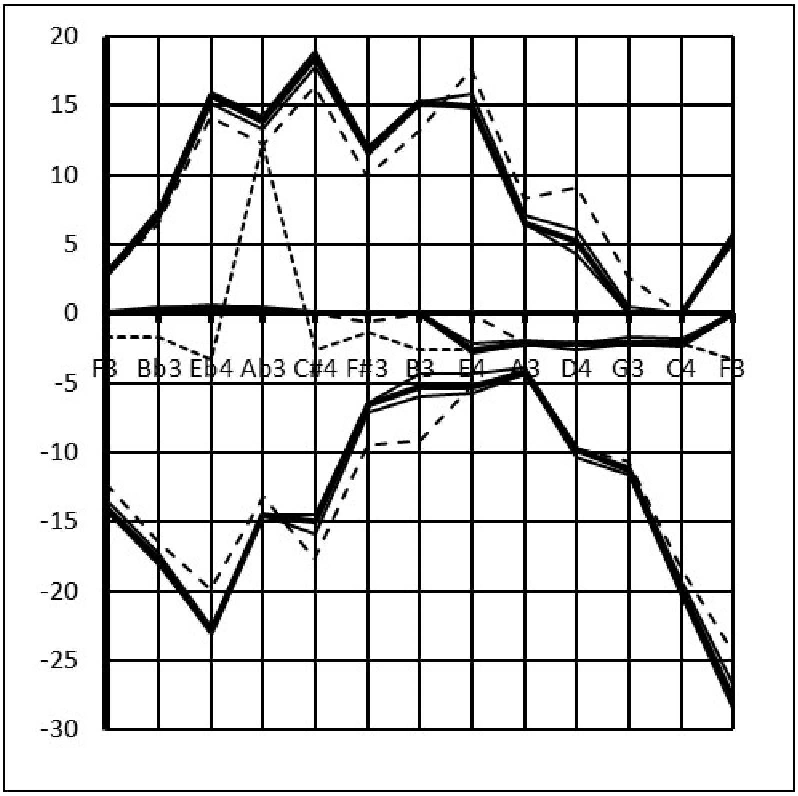

Fig 9 : Fifths and major thirds beat rate characteristics ***(within the F2-F3 scale)***
Slim lines : the minimal and maximal Bach beat rates ; ***Fat lines : BachdE***

and G can very probably be ***"THE" important precision check-points for the auditory tuner*** when setting this temperament.

This picture of the displayed Bach temperaments is remarkable.

It looks as if their tuning is derived from the meantone, maintaining two just major thirds, whereby the meantone diminished fifths on E*b*, B*b*, F, B, F#, C# and the wolf fifth on G# (A*b*)are enhanced to almost perfect ones. This is not surprising : the meantone temperament was the dominating one during the early Baroque, and Bach's musical education was based on the meantone (Kelletat, 1981 p. 21, lines 3 to 9).

This is in line also with the Kelletat/Werckmeister definition of well temperaments : "*...This temperament acts, while tied to given pitch ratios, as a thriftily tempered smoothing and extension of the meantone...*", and is in line also with historic comments that Bach recommended that (a number of) major thirds should be sharp (Kelletat 1981 p.51, footnote 69).

Figure 9 displays the beat rate calculated Kirnberger III also. It illustrates Kelletat's opinion that Bach may have applied Kirnberger III ***or another similar temperament***. Kirnberger III differs from Bach, mainly because of a pure fifth on E, inducing in turn some other remarkable differences, mainly on the major thirds on G and D.

After setting the just major thirds on C and G, and the pure fifths, the beat rate Bach tuning requires nothing more but the distribution of a very small impurity over three of the remaining fifths : G#, E*b*, B*b*.
The choice of one of Bach's compared temperaments does apparently not really import.

4 Bach's Tempered Meantone

4.1 Determination of the diatonic C-major scale

An ultimate observation of the Bach spirals, can give rise to doubts about the hypothesis that the fifths on F should be perfect. These fifths indeed, are associated with spirals too. The left F note contains only one scroll, the one on the right contains several scrolls, what could mean those fifths are not perfect.

This image, moreover, could suggest that tuning should be initiated within a F to F scale (from F3 to F4), the scrolls differences suggesting that the F note on the right is the upper note, having therefore a higher beat rate than the lower one, which is perfectly normal because of the interval equalling one octave.

Remains the question which properties to associate to the F note or its fifth. In line with the Jobin hypothesis, holding two just major thirds, C and G, it seems logical to suppose the F note could hold a just major third as well.

A just major third on F, combined with the C, G, D, A, E fifths properties (according the Bach–dE model) leads to the scale displayed at table 16.

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Bach =FCG=	Pitches	248,17	261,00	277,36	294,21	310,21	332,00	348,01	371,21	391,51	415,00	441,89	464,01

Table 16 : Bach =FCG= with almost equal fifths beat rate

All C-major diatonic notes pitches are identical to the corresponding meantone note pitches.

The beat rates are displayed at table 17l

	F3	Bb3	Eb3	Ab3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
T. maj.	0,00	4,71	13,78	13,90	22,98	13,77	16,82	14,99	6,52	5,21	0,00	0,00	0,00
Qui.	– 1,67	1,16	1,16	1,16	0,00	0,00	0,00	– 2,61	– 2,09	– 2,09	– 2,09	– 2,09	– 3,34
T. min.	– 17,23	– 20,66	– 25,23	– 14,50	– 14,99	– 6,52	– 5,21	– 5,21	– 4,17	– 4,17	– 8,88	– 17,95	– 34,47

Tableau 17 : Échelle BACH=FCG= ; beat rates

Moreover, the introduction of a just major third on the F note leads to a substantial diminution of the impurity of the minor third on the D4, to a level that is comparable, even better, than that of the three other minor thirds of C-major (on notes E4, A3, B3).

The fifths course is very remarkable, and displays ***an image reflecting exactly the hypotheses related to the Bach spirals*** : see figures 8 and 10. There is even more : the fifths also show some properties advocated by Sparschuh and Zapf, if one accepts indeed the condition that the fifths on Bb3, Eb3 and Ab3 can have a positive beat rate, instead of a negative one.

Figure 10 : Bach =FCG= ; course of fifths beat rate

The course of thirds and fifths beat rates is displayed at figure 11 (thin lines).

Two alternatives holding five or six diatonic C–major fifths that with best possible equality are mathematically worked out in appendices B–B3 and B–B4 (see also the "Cervoreille" (Brainear)", Calvet 2020). The pitches are almost identical to those obtained here, what means this model is close to a mathematical optimum.

Alternatives leading to ***best possible equality of major thirds and fifths impurities*** can also be worked out ; see appendices B–B5 and B–B6. Those alternatives lead to a significant reduction of the augmentation of A*b*, E*b* and B*b* fifths.

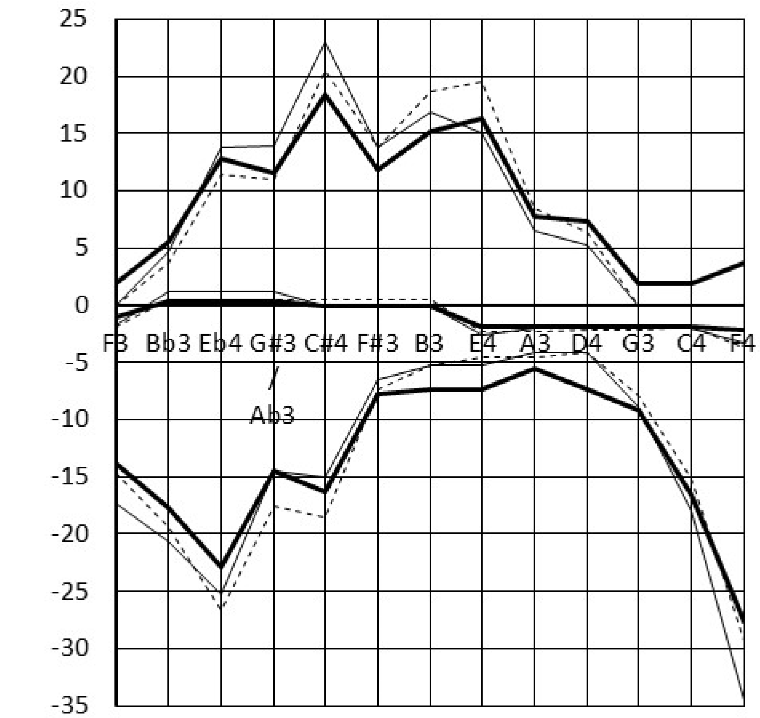

Figure 11 : Bach–2T≈ : bold lines : Appendix B–B5 version
Bach–=FCG= : thin line
dotted lines : equal impurity distribution for fifths on Bb3 to B3

Alternative scales based on interval ratios calculations, the "classical" calculation, are worked out for pure major thirds, as well as for major thirds and fifths with equal impurity ;see appendices

C–C2 to C–C4.

The appendices B–B7 and C–C5 offer beat rate and cent alternatives holding a minimum total impurity for major thirds and fifths.

Some of the obtained alternatives hold remarkable properties.
The B–B5 and B–B6 alternatives draw the attention (see the B5 course, fig 11, in bold lines). ***The B5 version holds a very remarkable and identical equality of 1.8362831858… beats/sec***. The B6 version is almost identical tot B5, with a slightly less remarkable equality.

An "equal beat" prerogative could be attributed to versions B5 and B6.

Could it be that these B5 and B6 characteristics, of almost equal beat rate of three major thirds and six fifths of the diatonic scale in C – major, lead to a confusion which has long reigned between the "well temperament" and the "equal temperament" concept (i.e. the confusion between "wohltemperiert" and "gleichschwebend") ?

Could it be that for Bach's "well temperament", there was therefore question of the "equal beat" discussed here, rather than the equal beat of all fifths in the cycle of fifths ?

Kellner (1977) was probablu among the firsts to work on mathematical determination of beat rate equality of fifths and thirds.

Table 18 displays all alternatives of this paper.

Jobin	2 pure major thirds (PMT) : five equal fifths (cent calculation)	Jobin
C	1 PMT : on C ; equal to meantone	Par. 3.1
dC	2 PMT : on C and G ; different fifth on C	Par. 3.2
dE	2 PMT : on C and G ; different fifth on E (equal to meantone)	Par. 3.2
2PT / 5F≈	2 PMT : on C and G ; five fifths with best equality of impurities	Appendix B–B1
2T ≈ 5F	Best possible equality of two major thirds and five fifths	Appendix B–B2
2T = 5F / cent	Best possible equality of two major thirds and five fifths (in cents)	Appendix C–C1
=FCG=	3 PMT : on F, C, G ; equal to meantone	Par. 4.1
3PT / 5F≈	3 PMT : on F, C, G ; five fifths with best equality of impurities	Appendix B–B3
3PT / 6F≈	3 PMT : sur F, C, G ; six fifths with best equality of impurities	Appendix B–B4
3T ≈ 5F	**Best possible equality of three major thirds and five fifths**	**Appendix B–B5**
3T ≈ 6F	**Best possible equality of three major thirds and fsix fifths**	**Appendix B–B6**
minimum	Lowest possible impurity for 6 fifths and 3 thirds	Appendix B–B7
3T / 5F≈/cent	3 PMT : on F, C, G ; and 5 fifths holding equal impurity (in cent)	Appendix C-C2
3TP /cent	5 fifths and 3 major thirds with equal impurity (in cent)	Appendix C-C3
3T = 6F/cent	6 fifths and 3 major thirds with almost equal impurity (in cent)	Appendix C-C4
Minimum / cent	Lowest possible impurity for 6 fifths and 3 thirds (in cent)	Appendix C-C5

Table 18 : Bach alternatives worked out in this paper

4.2 Determination of the fifths on the altered notes and on the B note

A proposed hypothesis on this subject might be controversial. The possibility for a better objective and rational alternative explanation may not be excluded.

The proposed Bach models in this paper lead to the requirement to hold ***AT LEAST ONE augmented fifth on the altered notes or the B note*** (the mean ratio of those fifths exceeds slightly the 1.50 ratio of perfect fifths)***. An extension to more diminished fifths on top of the six already obtained ones must therefore be avoided***, because this leads to a supplementary augmentation for the remaining fifths, what in turn leads to meantone characteristics "to be avoided" : excessively augmented fifths and harsh major thirds.

An even distribution of fifths impurities on altered notes and the B note, leads to beat rates displayed by the dotted lines on figure 11. This course corresponds to lesser quality of major thirds on D4, A3, E4, B3, but also better ones on B*b*3, E*b*4 et A*b*3. It appears the opposite is desired. A further analysis is required.

Based on the findings of figure 11, one could determinate an optimum for the major thirds on A3, E4 and B*b*3, combined with an optimal purity for the fifths on B3, and F#3 to B*b*3. The mathematical result leads to three just major thirds on A3, E4 et B*b*3. This result is unfortunately not acceptable : ***the concerned fifths also are diminishing, and therefore the leftover fifths and thirds will augment too much***.

An alternative approach can be worked out, by scouting for an optimal distribution of fifths impurities, ***whereby the fifths must be perfect or augmented***. To do so, the six fifths on B3, and F#3 to B*b*3 are calculated by means of the equations table 2. The calculation is done with all possible combinations of " *nq* " with n = "zero" or "one" as substitution for q_{B3} and $q_{F\#3}$ to q_{Bb3}, except the combination holding six zero's ; 63 combinations are possible (= $2^6 - 1$). The major thirds of permitted meantone keys (those on B*b*3, E*b*3, E4, A3, D4) are also calculated. The results are on display in appendix D.

Differing sorting and analyses on the tables of appendix D are possible. It can be observed that sorting based on the sum of the impurity of a fifth and the impurities of the major thirds on A3, E4 and B*b*3 leads to an absolute minimum (minimum minimorum) for the 111000 combination for "n" for $q_{Bb3,Eb4,Ab3,C\#4,F\#3,B3}$. Results are displayed on table 20, and the first row of table D3.

Number of enlargements	Sequence of enlargements : Bb3, Eb4, Ab3, C#4, F#3, B	fifths beat rate " q "	Major thirds beat rate			q+p_{E4} +p_{A3}+p_{D4}
			p_{E4}	p_{A3}	p_{D4}	
3	111000	1.16	14.99	6.52	5.21	27.88

Table 20 : result obtained in one sorting step, for a minimum of the sum q+E4+A3+D4

The herewith proposed hypothesis seems plausible, but leaves space for discussions.

The hypothesis supports mathematically the concept for which, after determination of the seven natural notes of the meantone, an optimal purity is desired for the allowed meantone keys holding sharp symbols (G–, D–, A–major), by installing perfect fifths on C#4, F#3 and G#3 (A*b*3), followed by an optimisation of the remaining fifths.

There exist quite some temperaments derived from the meantone, holding one or two augmented fifths, E*b*4 and A*b*3. Among others, not at least Rameau (1726), but also some more, like Marpurg (1752), Louet (1797), de Béthisy (1764), Vogel (1985), d'Alembert (1752), Legros (1972/75). A better major third on B*b*3 is obtained, if only two fifths are augmented.

The above findings concerning the notes on F#3, C#4, A*b*3, Eb4, B*b*3, are probably not of main importance, considering the over all properties of the here proposed Bach tuning schemes. ***Of main importance are the three major thirds of C–major, as already stated before.***

4.4 Conclusion concerning the Bach alternatives

One can define all above proposed alternatives as temperaments for which it is clear those are derived from the meantone, –all C-major diatonic notes are equal or almost identical, indeed, and for which the major thirds of permitted keys holding sharp symbols (#) should be the best possible, and with all of this combined with an optimisation of the fifths to be defined still.

The here given definition has a ***non determining*** consequence, that an almost equal diatonic fifths beat rate can be observed, and precise equal beat rates on the impure fifths on B*b*3, E*b*4, A*b*3.

5 Postamble

Famous historical temperaments, were defined based on interval beat rates instead of the more common application of interval ratios, or cent or comma deviations. It has become clear, that all those temperaments :

- are almost identical to their counterpart, commonly based on ratios, cents or comma's.
- require a tuning fork only, sometimes maybe a metronome, for easy auditory tuning

It is not unreasonable to assume that Baroque temperaments were conceived, based on auditory observation by interpreting musicians and auditory tuners, and could therefore have been based on optimisation of interval beat rates.

Post fact executed measurements by monochord at Baroque time, might at that time have lead to the commonly published definitions, based on cents, ratios or comma's, slightly differing from the actual installed one based on beat rates, because of unobserved very minor measuring errors (unobservable at that time !). Very minor measuring errors in cents are almost unavoidable, and the slight anharmonic structure and pitch sliding of the real physical vibrators of musical instruments also contributes to measuring uncertainties.

All above findings are valid in particular, for the Bach tuning according the Jobin proposal and the a number of hypotheses proposed in this paper.

The exceptional dexterity and speed with which Bach could auditory tune a clavichord (Kelletat, 1981, p. 52-53; Forkel, p.17) allows to assume that he only tuned by the ear, probably based on observable beat rates. Bach was not interested in the measurement of interval ratios (Forkel, p. 39).

Based on all above arguments, it seems reasonable to assume that the spirals on the score of "Das wohltemperirte Clavier" may be associated to purity requirements concerning beats and beat rates of fifths, indeed, associated also to just major thirds on C and G, and F too, this is in support still, of Jobin's hypothesis based on the significance of those fifths measured by commonly applied equality significance of ratios, but still in support also of the here elaborated alternative beat rate Bach temperaments.

IT IS NOT THE STRICT MATHEMATICAL EXACTITUDE OF FIFTHS THAT PREVAILS, to define a "BACH-TEMPERAMENT",… but that what prevails in this indeed, is the "AUDITORY EQUALITY JUDGMENT" of the interpreting musicians and auditory tuners, their "brainear (cervoreille)", leading to JUST OR ALMOST JUST MAJOR THIRDS on C and G, and also F.

Conclusion

- It are the beat rates that import to performing musicians and auditory tuners
- Beat rate characteristics of historical temperaments are probably their determining factor
- Beat rate characteristics should probably deserve much more attention in musicology, and calculation of temperaments, especially for the Baroque period

- Taking into account the achievable auditory tuning precisions, it must be seen as a quasi certainty that Jobin and the proposed and almost equivalent beat rate Bach temperaments are valid hypotheses.

Dedication

This paper is dedicated to all classic musicians and auditory tuners.
Their sensitive musical ears offer to our world all the best of the most universal and most beautiful of all languages : ***MUSIC***.

Acknowledgment

I want to express my most sincere feelings of gratitude towards Amiot E., Calvet A. Jobin E and Paintoux T.
Their open attitude allowed for quite intense exchange of ideas, that enabled further understanding and evolvement of insights in musical temperament and tuning matters, enabling the development of the ideas and concepts expressed in this paper.

Thanks to my daughter Hilde : she drew my attention to investigate on what musicians want (diatonic interval purity) and not on what might be someone's preferred musical temperament.

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Zapf 2001: informal publication

Appendix A Beat rate recalculation of some historic temperaments

Table A1 displays the applied comma divisions (q/n) and the obtained beat rates, applying the formulas of tables 2 and 4. It also displays the RMS−Δ−cent of the pitch deviations from the “classic” temperament. See also : Calvet 2020.
Table A2 displays the obtained note pitches.

	Auditory tuning instructions (figures in beats/sec.)	RMS Δ-cent
Bendeler III 1690	q/3= − 1.14 on C, G, E, G#	1.09
Kellner 1976	q/5= − 0.92 on C, G, D, A, B	0.37
Lehman 2005	q/6= − 0.63 on F, C, G, D, A | q/12= − 0.31 on C#, G#, E*b*, Bd	1.22
Mercadier 1788	q/16= − 0.28 on E, B, F#, C# | q/12= − 0.37 on F | q/6= − 0.74 on C, G, D	0.31
Neidhardt 1 1732	q/6= − 0.78 on C, G, D, A | q/12= − 0.39 on E, B, G#, E*b*	0.27
Neidhardt 2 1732	q/6= − 0.73 on C, G, D | q/12= − 0.37 on F, A, B, F#, C#, B*b*	0.34
Neidhardt 3 1723	q/6= − 0.77 on C, G, D | q/12= − 0.39 on A, B, F#, C#, E*b*, B*b*	0.37
Neidhardt 4 1732	q/3= − 1.15 on D, A | q/6= − 0.77 on G | q/12= − 0.38 on C, B, B*b*, C#	0.48
Sorge 1744	q/6= − 0.81 on C, G, D, E | q/12= − 0.40 on B, F#, E*b*, B*b*	0.50
Sorge 1758	q/6= − 0.77 on C, G, D | q/12= − 0.39 on A, B, F#, C#, E*b*, B*b*	0.78
Stanhope 1806	(synt. comma)/3= − 1.35 on G, D, A | (schism. comma)= − 0.22 on B, E*b*	0.57
Vogel 1975	q/7= − 0.98 on F, C, G, A, D, E, C# | q/7 = 0.98 on F#, B*b*	1.41
Werckmeister VI 1691	4q/7= − 2.18 on G | 2q/7= − 1.09 on F# | q/7= − 0.55 on C, B, B*b* | q/7= +0.55 on D, G#	0.56

Table A1 : Tuning instructions and RMS Δ-cent of note pitch deviations from conventional values

	C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
Bendeler III 1690	247.60	261.73	276.67	293.45	311.25	330.13	348.97	370.14	392.59	415.00	440.17	465.30
	247.71	261.97	276.67	293.58	311.25	330.27	349.30	370.41	392.96	415.00	440.36	465.73
Kellner 1976	247.93	261.20	277.42	293.85	310.41	330.58	348.26	370.89	391.80	415.00	440.77	465.61
	247.89	261.16	277.28	293.80	310.33	330.53	348.21	370.93	391.73	415.00	440.70	465.50
Lehman 2005	247.60	262.02	277.29	294.11	310.55	330.88	349.37	370.56	392.59	415.00	440.67	465.82
	247.27	262.09	277.09	294.30	310.62	349.45	349.45	370.28	392.82	415.00	441.13	465.93
Mercadier 1788	247.60	261.36	277.29	293.78	310.55	330.50	348.78	370.56	391.71	415.00	440.67	465.43
	247.52	261.34	277.16	293.80	310.51	330.53	348.83	370.54	391.74	415.00	440.70	465.48
Neidhardt 1 1732	247.60	261.43	277.29	293.78	310.55	330.13	348.58	370.56	392.15	415.00	440.17	465.30
	247.61	261.44	277.19	293.73	310.47	330.15	348.59	370.63	392.16	415.00	440.20	465.31
Neidhardt 2 1732	247.60	261.73	277.29	294.11	310.90	330.50	349.37	370.56	392.15	415.00	441.17	466.35
	247.50	261.67	277.16	294.10	310.88	330.49	349.38	370.52	392.13	415.00	441.15	466.32
Neidhardt 3 1723	247.60	261.73	277.29	294.11	310.90	330.13	349.37	370.56	392.15	415.00	440.67	466.35
	247.58	261.62	277.18	294.03	310.86	330.11	349.34	370.60	392.04	415.00	440.66	466.30
Neidhardt 4 1732	247.60	261.43	277.61	293.78	310.20	330.13	348.58	370.98	391.71	415.00	440.67	465.30
	247.55	261.35	277.44	293.74	310.10	350.33	348.47	370.94	391.65	415.00	440.60	465.14
Sorge 1744	247.60	261.43	277.29	294.11	311.25	330.13	348.97	370.56	392.15	415.00	440.67	465.82
	247.65	261.46	277.20	294.14	311.25	330.21	349.15	370.68	392.19	415.00	440.81	466.07
Sorge 1758	247.60	261.73	277.29	294.11	310.90	330.13	349.37	370.56	392.59	415.00	440.67	466.35
	247.58	261.62	277.18	294.03	310.86	330.11	349.34	370.60	392.04	415.00	440.66	466.30
Stanhope 1806	247.97	261.38	277.81	294.06	309.96	330.63	348.51	371.96	392.08	415.00	440.84	464.95
	247.92	261.32	277.56	293.98	309.90	330.56	348.42	371.88	391.97	415.00	440.75	464.85
Vogel 1975	248.05	261.97	277.67	293.98	310.48	331.55	348.42	371.15	391.97	415.00	440.97	464.56
	248.04	262.22	277.32	294.26	310.27	332.02	348.31	371.07	392.35	415.00	441.39	464.42
Werckmeister VI	247.84	261.10	276.13	294.30	311.25	330.45	349.48	371.04	391.64	415.00	441.45	466.88
	247.91	261.12	276.30	294.30	311.25	330.54	349.61	371.32	391.67	415.00	441.45	466.88

Table A2 : Comparison of temperament pitches : the lower rows are the beat rate calculated versions.

Probably many more temperaments could be redefined or recalculated (see for example : Calvet, Jedrzejewski F. 2002).

Appendix B Minimisation of interval impurity differences.
The « Brainear » (« Le Cervoreille » ; Calvet 2020)

Equal interval impurities can be set with relative low effort by any professional auditive keyboard tuner. This strict equality of interval impurities, such as calculated for a number of historical temperaments, is however not always easy nor possible.

This problem occurs, for example, if a group of fifths should contain more than one just major third, such as encountered for the Bach hypotheses. A strict mathematical equality is not achievable in that case. The experience and professionalism of the tuner are of primordial importance under this peculiar condition. Besides the tuners or musicians fine ear, intervenes fundamentally also his mental (musical) judgment of the tuning : ***his "Brainear"*** (Cervoreille, Calvet 2020).

An optimisation of interval impurity, can be calculated by determination of the ***minimum of their impurity differences***, rather than their strict mathematic equality.

The mean beat rate of m major thirds and n fifths***, calculated based on their absolute values*** (positive thus), equals (q_{Note} and $_{\text{pNote}}$: see tables 2 and 4) :

$beat_{mean} = \sum \frac{-q_n + p_m}{m+n}$; the note interval impurity deviation from the mean is :

$\Delta q_{Note} = q_{Note} + \sum \frac{-q_n + p_m}{m+n}$ for the fifths ; and

$\Delta p_{Note} = -p_{Note} + \sum \frac{-q_n + p_m}{m+n}$ for the thirds

And therefore the following expression should be minimised : $S_n = \sum [(\Delta q_{Note})^2 + (\Delta p_{Note})^2]$

This can be simplified to the expression below, If fifths only have to be minimised :

$$S_n \propto \frac{n-1}{2} \sum_{m=1}^{n} q_m^2 - \sum_{(a;b)=(1\to n)}^{a\neq b} q_a q_b$$

This expression can be minimised by calculation of its partial derivatives set to zero, followed by solving the thus obtained equations.

If fifths ***and*** thirds have to be optimised, it is necessary to make the full detailed calculation.

To do so, ***the impurities must be independent variables***. The several impurities must therefore first be substituted by their expressions in function of the constituting notes (see tables 2 and 4).

Appendix B1 Just major thirds on C and G,
requiring optimal equalisation of the concerned fifths on C, G, D, A, E

In order to comply with the condition holding just major thirds on C4 and G3, it is necessary to substitute note E4 by its value 5D4/4 and note B3 by its value 5G3/4. The sum of purity deviations of the fifths on C, G, D, A, E will therefore depend on three notes only : C, G, D. The obtained result is :

$$\sum \Delta q_{notes}^2 \propto 64.25C_4^2 + 107G_3^2 + 32D_4^2 + 62A_3^2 \\ -128.25C_4G_3 - 4.25C_4D_4 - 33.25C_4A_3 - 24G_3D_4 - 6G_3A_3 - 59D_4A_3$$

The partial derivatives lead to following equations :

	C4	G3	D4	=	A3
C4	128.5	– 128.25	– 4.25	=	33.25
G3	– 128.25	214	– 24	=	6
D4	– 4.25	– 24	64	=	59

Table B1 : Equations for note pitch calculations

The E4 and B3 note are obtained based on just major thirds on C4 and G3.
The F#3, C#4, G#3/A*b*3 notes are calculated based on perfect fifths on B3 ; F3 as a fifth below C4.
The B*b*3, E*b*4 notes are calculated for equal beating between A*b*3 and F3.
Obtained solutions :

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
2PT / ≈5F≈	Pitches	248.14	261.05	277.38	293.84	310.18	330.86	348.07	371.27	391.58	415.00	440.93	464.09

Table B1 : BACH–2PT / ≈5F≈ scale ; note pitches, for minimal beat rate differences of optimised fifths

The obtained beat rates are on display in table B2.

	F3	B*b*3	E*b*3	A*b*3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Maj. Th.	2.86	7.21	15.88	13.62	18.16	11.68	15.15	15.43	6.71	5.37	0.00	0.00	5.72
Fifths	0.00	0.32	0.32	0.32	0.00	0.00	0.00	–2.35	– 2.15	– 2.14	– 2.15	– 1.88	0.00
min. th.	– 13.62	– 17.52	– 22.72	– 14.50	– 15.43	– 6.71	– 5.37	– 4.69	– 4.29	– 10.01	– 11.51	– 19.63	– 27.24

Table B2 : BACH–2PT / ≈5F≈ ; beat rates

Appendix B2 Equality of impurities on major thirds on F, C et G, and fifths on C, G, D, A, E

An equalization of the beat rates is possible, for fifths on C, G, D, A, and the major third on C. In fact, we obtain five linear equations with five unknowns: the unknowns Do, Sol, Re, Mi, and "Beat $Battement1 = 3G3 - 2C4 = 3D4 - 4G3 = 3A3 - 2D4 = 3E4 - 4A3 = 5C4 - 4E4$

$$Beat1 = 3G3 - 2C4 = 3D4 - 4G3 = 3A3 - 2D4 = 3E4 - 4A3 = 5C4 - 4E4$$

We can add a sixth condition leading to the unknown fifth note B3, based on the same equal beat for the fifth on E: $Beat1 = 3B3 - 2\,E4$

It can be verified, that a mathematical chance leads, WITHOUT ANY MORE, also to a major third with equal beat on G3, which makes that an addition of a seventh equation to satisfy this condition is not necessary.

Applying the obtained solutions, the F3 note can be finally be calculated bases on a pure fifth under the C-note: $3F3 - 2C4 = 0$. All the above conditions lead to:

$$Beat1 = \frac{C4}{135} = \frac{D4}{151} = \frac{E4}{169} = \frac{3 \times F3}{270} = \frac{G3}{101} = \frac{A3}{113} = \frac{2 \times B3}{253}$$

Notes F3, F#3, C#4, G#3/Ab3 are obtained, calculating perfect fifths :

$$0 = 3F3 - 2C4 = 3B3 - 4F\#3 = 3F\#3 - 2C\#4 = 3C\#4 - 4G\#3$$

Notes Eb4 and Bb3 are obtained calculating equal beat rates for the fifths on A*b*3, E*b*3, B*b*3 :

$$Battement = 3Ab3 - 2Eb4 = 3Eb4 - 4Bb3 = 3Bb3 - 4F3$$

Obtained solutions :

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
2T ≈5F	Pitches	247.90	261.33	277.28	293.92	310.33	330.53	348.43	370.93	391.99	415.00	440.81	464.58

Tableau B–2T ≈5F scale ; note pitches for least inequality of beat rate impurities on fifths and concerned thirds

The beat rates are on display in table B6.

	F3	B*b*3	E*b*3	A*b*3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Maj. Th	3.67	7.10	14.12	11.62	15.49	10.52	14.23	16.30	7.80	7.35	1.84	1.84	7.35
Fifths	0.00	– 0.15	– 0.15	– 0.15	0.00	0.00	0.00	– 1.84	– 1.84	– 1.84	– 1.84	– 1.84	0.00
min. th.	– 11.62	– 15.79	– 21.34	– 14.52	– 16.30	– 7.80	– 7.35	– 7.35	– 5.51	– 11.02	– 10.77	– 17.80	– 23.24

Tableau B6 : Échelle BACH–2T ≈ 5F ; battements d'harmoniques

Appendix B3 Just major thirds on F, C and G and optimal equalisation of beat rates of concerned fifths on C, G, D, A end E

Results can easily be obtained using the solution of appendix B1, but by setting a just major third on F, instead of a perfect fifth. Obtained solution :

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
3PT / 5F≈	Pitches	248.14	261.05	277.38	294.16	310.18	332.00	348.07	371.27	391.58	415.00	441.95	464.09

Table B7 : BACH–3PT / 5F≈ scale ; note pitches for optimised equalisation of fifths beat rate

The obtained beat rates are on display in table B8.

	F3	B*b*3	E*b*3	A*b*3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Maj. Th.	0.00	4.65	14.30	13.62	22.74	13.73	16.40	15.43	6.71	5.37	0.00	0.00	0.00
Fifths	– 1.72	1.07	1.43	0.95	0.00	0.00	0.00	– 2.35	– 2.15	– 2.14	– 2.15	– 1.88	– 3.43
min. th.	– 17.06	– 20.60	– 24.61	– 14.50	– 15.43	– 6.71	– 5.37	– 4.69	– 4.29	– 4.29	– 8.94	– 18.06	– 34.11

Table B8 : BACH–3PT / 5F≈ scale ; beat rates

Appendix B4 Just major thirds on F, C et G, combined with an optimals equalisation of concerned fifths on F, C, G, D, A, E

Application of the same procedure as for appendix B1, leads to following sum :

$$97.40625C_4^2 + 132G_3^2 + 38.5D_4^2 + 91.3A_3^2 - 171C_4G_3 - 2.25C_4D_4 - 76.95C_4A_3 + 8.4G_3A_3 - 73.4D_4A_3$$

Leading to the equations :

	C4	G3	D4	=	A3
C4	194.8125	– 171	– 2.25	=	76.95
G3	– 171	264	– 30	=	– 8.4
D4	– 2.25	– 30	77	=	73.4

Table B9 : Equations leading to the pitches of C4, G3, D4

Obtained temperament pitches :

		C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
3TP / 6Q	Pitches	248,09	261,01	277,36	294,13	310,11	332,00	348,01	371,22	391,52	415,00	441,93	464,02

Tableau B10 : Échelle Bach–3TP / 6F scale : à quasi égalité de battement harmonique des SIX quintes optimalisée

Obtained beat rates :

	F3	B*b*3	E*b*3	A*b*3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4

Maj. Th	0.00	4.63	14.22	13.55	22.94	13.82	16.46	15.53	6.54	5.25	0.00	0.00	0.00
Fifths	– 1.83	1.11	1.47	0.98	0.00	0.00	0.00	– 2.28	– 2.29	– 2.09	– 2.10	– 1.82	– 3.66
min. th.	– 17.21	– 20.73	– 24.69	– 14.50	– 15.53	– 6.54	– 5.25	– 4.56	– 4.57	– 4.18	– 8.83	– 17.87	– 34.42

Tableau B11 : Bach–3TP / 6F scale ; beat rates

Appendix B5 Maximal equalisation of impurities on major thirds on F, C et G, and fifths on C, G, D, A, E

We can use the solution of Appendix B – B2, except that the note F3 is calculated based on a third F3 – A3 major third, rather than based on a perfectF3 – C4 fifth. We thus obtain as an equal beat:

$$Battement1 = \frac{C4}{135} = \frac{D4}{151} = \frac{E4}{169} = \frac{5 \times F3}{451} = \frac{G3}{101} = \frac{A3}{113} = \frac{2 \times B3}{253}$$

The obtained pitches are :

C4	C#4	D4	Eb4	E4	F4	F#4	G4	G#4	A4	Bb4	B4
247.90	261.33	277.28	294.17	310.33	331.27	348.43	370.93	391.99	415.00	441.44	464.58

Table B13 : BACH–3T ≈5F≈ scale ; note pitches for optimised beat rate equality of five fifths and concerned thirds

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	1.84	5.51	12.85	11.62	18.43	11.80	15.25	16.30	7.80	7.35	1.84	1.84	1.84
Fifths	– 1.10	0.37	0.37	0.37	0.00	0.00	0.00	– 1.84	– 1.84	– 1.84	– 1.84	– 1.84	– 1.10
Minor thirds	– 13.82	– 17.70	– 22.87	– 14.52	– 16.30	– 7.80	– 7.35	– 7.35	– 5.51	– 7.35	– 9.18	– 16.52	– 13.82

Table B14 : BACH–3T ≈5F scale ; beat rates

Appendx B6 Maximal equalisation of the beat rate impurities of the major thirds on F, C et G, and the fifths on F, C, G, D, A, E

The sum of the squares of the impurities corresponds to :

$$\begin{aligned} 81 \times \sum \Delta^2 = {} & 2718F_3^2 + 2934C_4^2 + 3726G_3^2 + 1044D_4^2 + 3240A_3^2 + 2124E_4^2 + 2592B_3^2 \\ & -1116F_3C_4 - 216F_3G_3 + 36F_3D_4 - 3132F_3A_3 + 180F_3E_4 \\ & -2376C_4G_3 + 72C_4D_4 + 216C_4A_3 - 2880C_4E_4 \\ & -864G_3D_4 + 324G_3A_3 + 540G_3E_4 - 3240G_3B_3 \\ & -1998D_4A_3 - 90D_4E_4 \quad - 1242A_3E_4 \quad - 1944E_4B_3 \end{aligned}$$

The partial derivatives set to zero lead to the equations of table B15 :

	F3	C4	G3	D4	E4	B3	=	A3
F3	5436	– 1116	– 216	36	180	0	=	3132
C4	– 1116	5868	– 2376	72	– 2880	0	=	– 216
G3	– 216	– 2376	7452	– 864	540	– 3240	=	– 324
D4	36	72	– 864	2088	– 90	0	=	1998
E4	180	-2880	540	-90	4248	-1944	=	1242
B3	0	0	– 3240	0	– 1944	5184	=	0

Table B15 : definition of the diatonic notess

The notes F#3, C#4, G#3/Ab3, are calculated based on perfect fifths :

$$0 = 3B3 - 4F\#3 = 3F\#3 - 2C\#4 = 3C\#4 - 4G\#3$$

The notes Eb4 and Bb3 are obtained setting an equal beat rate for fifths on Ab3, Eb3, Bb3 :

$$Beat = 3Ab3 - 2Eb4 = 3Eb4 - 4Bb3 = 3Bb3 - 4F3$$

The obtained pitches are :

C4	C#4	D4	Eb4	E4	F4	F#4	G4	G#4	A4	Bb4	B4
247,82	261,26	277,26	294,17	310,24	331,38	348,35	370,86	391,89	415,00	441,50	464,47

Table B16 : BACH–3T ≈6F≈ scale ; note pitches for optimised beat rate equality of six fifths and concerned thirds

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	1,56	5,27	12,60	11,54	19,20	12,13	15,51	16,39	7,55	7,12	1,78	1,86	3,11
Fifths	– 1,43	0,50	0,50	0,50	0,00	0,00	0,00	– 1,78	– 2,03	– 1,77	– 1,78	– 1,73	– 2,86
Minor thirds	– 14,40	– 18,20	– 23,27	– 14,51	– 16,39	– 7,55	– 7,12	– 7,12	– 5,92	– 6,65	– 8,82	– 16,06	– 28,80

Table B17 : BACH–3T ≈6F≈ scale ; beat rates

Appendix B7 Beat rate impurity minimisation of major thirds on F, C et G, and fifths on F, C, G, D, A, E

A beat rate impurity minimisation of major thirds on F, C et G, and fifths on F, C, G, D, A, E can be obtained by calculating the minimum of their sum of beat rate squares, defined by the equations of tables 2 and 4. Setting the partial derivatives of this sum to zero, leads to the equations of table 18 :

	F3	C4	G3	D4	E4	B3	=	A3
F3	68	-12	0	0	0	0	=	40
C4	-12	76	-24	0	-40	0	=	0
G3	0	-24	100	-12	0	-40	=	0
D4	0	0	-12	26	0	0	=	24
E4	0	-40	0	0	58	-24	=	12
B3	68	-12	0	0	0	0	=	40

Table B18 : Calculation of note pitches leading to a minimal impurity of the diatonic C–major fifths and major thirds

The obtained note pitches are :

C4	C#4	D4	Eb4	E4	F4	F#4	G4	G#4	A4	Bb4	B4
248.23	261.63	277.33	294.51	310.36	331.73	348.84	371.76	392.44	415.00	442.04	248.23

Table B19 : BACH –minimum scale ;
note pitches for minimal beat rate impurity of fifths and major thirds of the diatonic C–major scale

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	0.68	4.22	14.48	11.83	18.77	11.98	15.25	17.98	9.02	8.71	0.84	0.26	1.35
Fifths	– 1.13	0.40	0.54	0.36	0.00	0.00	0.00	– 0.84	– 1.78	– 1.99	– 2.98	– 1.19	– 2.25
Minor thirds	– 14.08	– 17.97	– 22.88	– 14.53	– 17.98	– 9.02	– 8.71	– 3.36	– 3.83	– 5.32	– 10.18	– 16.85	– 28.16

Table B20 : BACH–minimum scale ; beat rates

Appendix C Ratio based calculations

Appendix C1 Bach hypothesis based on ratio calculations, holding two optimised major thirds, with equal impurity of diatonic major thirds and fifths

The scale holds five diminished fifths that build two slightly enlarged major thirds, and has four perfect fifths.

Let « R » be the equal impurity ratio of fifths and major thirds.

This ratio for four fifths, brought back to an octave, builds a major thirds with ratio : $\left(\frac{3}{2}R\right)^4 \times \frac{1}{4}$

The corresponding major third ratio is : $\frac{5}{4 \times R}$

Equality of both expressions leads to : $R = \sqrt[5]{\left(\frac{5 \times 16}{81}\right)} = 0.99752 \ldots$

The « S » impurity, of fifths between Ab and F can be defined the same way : $S = \frac{2}{3} \times \sqrt[3]{\frac{4F}{Ab}}$

The obtained pitches are :

C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
247.77	261.32	277.35	293.87	310.48	330.35	348.42	370.73	391.97	415.00	440.64	464.56

Table C1 : Bach–2T = 5F / cent scale ;note pitches

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	4.11	7.82	13.56	11.13	14.84	10.22	14.08	15.50	7.76	6.91	2.31	3.08	8.23
Fifthss	0.00	– 0.25	– 0.33	– 0.22	0.00	0.00	0.00	– 2.31	– 1.54	– 2.06	– 1.38	– 1.84	0.00
minor thirds	– 11.13	– 15.34	– 21.11	– 14.52	– 15.50	– 7.76	– 6.91	– 9.23	– 6.17	– 12.36	– 10.58	– 17.25	– 22.26

Table C2 : Bach–2T = 5F / cent scale ; beat rates

Observation : the equal impurity cannot be observed by evaluation of beat rates. A pitch measuring instrument is required to tune this temperament.

Appendix C2 Bach hypothesis, holding just major thirds on F, C, G and five fifths with equal impurity, based on ratio calculations

This version equals the C1 version, except for a just major third on F, instead of a perfect fifth.
The pitches, below in table C3 are obtained :

C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
247.77	261.32	277.35	294.36	310.48	332.00	348.42	370.73	391.97	415.00	442.10	464.56

Table C3 : BACH–1TP 2T = 5F/cent scale ; pitches

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	0.00	4.17	11.12	11.13	21.42	13.15	16.02	15.50	7.76	6.91	2.31	3.08	0.00
Fifths	– 2.47	0.85	1.13	0.75	0.00	0.00	0.00	– 2.31	– 1.54	– 2.06	– 1.38	– 1.84	– 4.94
minor thirds	– 16.07	– 19.72	– 24.04	– 14.52	– 15.50	– 7.76	– 6.91	– 9.23	– 6.17	– 4.13	– 6.93	– 14.81	– 32.13

Table C4 : BACH–1TP 2T = 5F/cent scale ; beat rates

Appendix C3 Bach hypotheses, holding three just major thirds and six fifths with equal impurity, based on ratio calculations

The calculation of major thirds and fifths requires the same formulas as for appendis C1.
The following pitches are obtained :

C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
248.23	260.99	277.53	294.11	310.28	332.00	347.99	371.19	391.49	415.00	441.92	463.98

Table C5 : BACH–3TP / 6F/cent scale ; pitches

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	0.00	5.32	14.19	14.20	23.04	13.86	16.49	14.52	6.46	4.32	0.00	0.00	0.00
Fifths	– 1.54	1.12	1.50	1.00	0.00	0.00	0.00	– 2.89	– 1.93	– 2.58	– 1.73	– 2.31	– 3.09
minor thirds	– 17.28	– 20.80	– 24.74	– 14.50	– 14.52	– 6.46	– 4.32	– 5.77	– 3.86	– 5.16	– 8.77	– 18.80	– 34.57

Table C6 : BACH– 3TP / 6F/cent scale ; beat rates

Appendix C4 Bach hypotheses, holding equal impurity for three major thirds and six fifths, based on ratio calculations

The calculation of major thirds and fifths requires the same formulas as for appendis C1.
The following pitches are obtained :

C	C#	D	E*b*	E	F	F#	G	G#	A	B*b*	B
247.77	261.32	277.35	294.11	310.48	331.18	348.42	370.73	391.97	415.00	441.37	464.56

Table C7 : BACH– 3T = 6F/cent scale; pitches

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	2.06	6.00	12.34	11.13	18.13	11.69	15.05	15.50	7.76	6.91	2.31	3.08	4.12
Fifths	– 1.23	0.30	0.40	0.27	0.00	0.00	0.00	– 2.31	– 1.54	– 2.06	– 1.38	– 1.84	– 2.47
minor thirds	– 13.60	– 17.53	– 22.57	– 14.52	– 15.50	– 7.76	– 6.91	– 9.23	– 6.17	– 8.25	– 8.76	– 16.03	– 27.19

Table C8 : BACH– 3T = 6F/cent scale; beat rates

Appendix C5 Minimisation of the F, C and G major thirds, and F, C, G, D, A, E fifths, impurities, calculated based on cent impurity values

A minimisation of the F, C and G major thirds, and F, C, G, D, A, E fifths, impurities, is obtained by determination of the minimum of the sum of their squares, defined in cents.
The obtained partial derivatives of this sum, set to zero, are displayed in table C9 :

	logF3	logC4	logG3	logD4	logE4	logB3	=	
logF3	2	-1	0	0	0	0	=	logA3+3+log3-log5+
logC4	-1	3	-1	0	-1	0	=	3-log5
logG3	0	-1	3	-1	0	-1	=	1-log5
logD4	0	0	-1	2	0	0	=	logA3+1
logE4	0	-1	0	0	3	-1	=	logA3-1+log5
logB3	0	0	-1	0	-1	2	=	-4+log3+log5

Table C9 : Note pitch calculation, leading to minimal impurity of major thirds and fifths of the diatonic C–major

The obtained pitches are :

C4	C#4	D4	Eb4	E4	F4	F#4	G4	G#4	A4	Bb4	B4
248.27	261.66	277.68	294.47	310.57	331.52	348.88	371.59	392.49	415.00	441.86	465.17

Table C10 : BACH–minimum/cent scale ;
note pitches for a minimal impurity of the major thirds an fifths of the diatonic C–major

Obtained beat rates :

	F3	Bb3	Eb4	G#3	C#4	F#3	B3	E4	A3	D4	G3	C4	F4
Major Thirds	1.21	6.06	14.02	11.87	17.76	11.53	14.95	17.11	9.14	7.12	1.36	0.91	2.42
Fifths	– 0.73	0.24	0.31	0.21	0.00	0.00	0.00	– 1.36	– 1.36	– 3.04	– 2.03	– 1.63	– 1.45
Minor thirds	– 13.32	– 17.29	– 22.43	– 14.54	– 17.11	– 9.14	– 7.12	– 5.44	– 3.63	– 8.50	– 10.12	– 17.28	– 26.65

Table C11 : BACH–minimum/cent scale ; beat rates

Appendix D

Beat rates of the major thirds that are normally just for the meantone, calculated in function of the impurities distribution of the fifths on the altered notes and on the B note.

B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
0	0	1	0	0	0	2.96	2,78	9,27	14,99	6,52	5,21	29,68
1	0	0	0	0	0	3.33	8,34	16,68	14,99	6,52	5,21	30,05
0	1	0	0	0	0	4.44	2,78	16,68	14,99	6,52	5,21	31,16
0	0	0	1	0	0	3.95	2,78	9,27	22,89	6,52	5,21	38,57
0	0	0	0	1	0	2.63	2,78	9,27	22,89	11,78	5,21	42,52
0	0	0	0	0	1	3.51	2,78	9,27	22,89	11,78	12,24	50,42

Table D1 : Results for one enlarged fifth ; sorted for a best value of $q + p_{E4} + p_{A3} + p_{D4}$

B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
1	0	1	0	0	0	1,57	5,39	12,75	14,99	6,52	5,21	28,29
0	1	1	0	0	0	1,78	2,78	12,23	14,99	6,52	5,21	28,50
1	1	0	0	0	0	1,90	5,95	16,68	14,99	6,52	5,21	28,62
0	0	1	1	0	0	1,69	2,78	9,27	18,37	6,52	5,21	31,80
1	0	0	1	0	0	1,81	5,79	13,29	18,60	6,52	5,21	32,14
0	1	0	1	0	0	2,09	2,78	12,75	19,17	6,52	5,21	32,99
0	0	1	0	1	0	1,39	2,78	9,27	19,17	9,31	5,21	35,08
1	0	0	0	1	0	1,47	5,23	12,54	19,40	9,46	5,21	35,55
0	1	0	0	1	0	1,65	2,78	12,02	19,95	9,82	5,21	36,64
0	0	1	0	0	1	1,61	2,78	9,27	18,60	8,93	8,43	37,57
1	0	0	0	0	1	1,71	5,63	13,07	18,84	9,08	8,63	38,26
0	0	0	1	1	0	1,58	2,78	9,27	22,89	9,68	5,21	39,36
0	1	0	0	0	1	1,96	2,78	12,54	19,40	9,46	9,14	39,96
0	0	0	1	0	1	1,86	2,78	9,27	22,89	9,31	8,93	42,99
0	0	0	0	1	1	1,50	2,78	9,27	22,89	11,78	8,22	44,40

Table D2 : Results for two enlarged fifths ; sorted for a best value of $q + p_{E4} + p_{A3} + p_{D4}$

B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
1	1	1	0	0	0	1,16	4,71	13,78	14,99	6,52	5,21	27,88
1	1	0	1	0	0	1,12	4,65	11,76	17,23	6,52	5,21	30,09
1	0	1	1	0	0	1,23	2,78	11,31	17,44	6,52	5,21	30,40
0	1	1	1	0	0	1,29	4,92	14,27	17,56	6,52	5,21	30,57
1	1	0	0	1	0	0,98	4,42	11,45	17,94	8,48	5,21	32,62
1	0	1	0	1	0	1,06	2,78	11,04	18,17	8,64	5,21	33,09
0	1	1	0	1	0	1,11	4,62	13,57	18,30	8,73	5,21	33,35
1	0	0	1	1	0	1,08	4,59	11,68	17,43	8,14	7,38	34,04
0	1	0	1	1	0	1,18	2,78	11,24	17,64	8,29	7,57	34,69
0	0	1	1	1	0	1,03	2,78	9,27	20,14	8,58	5,21	34,96
1	1	0	0	0	1	1,23	4,84	14,07	17,77	8,37	7,68	35,05
1	0	1	0	0	1	1,07	4,57	11,65	20,35	8,66	5,21	35,29
0	1	1	0	0	1	1,17	2,78	11,21	20,82	8,85	5,21	36,04
1	0	0	1	0	1	1,14	2,78	9,27	19,84	8,23	7,50	36,71
0	1	0	1	0	1	1,19	4,77	11,92	20,06	8,31	7,60	37,16
0	0	1	1	0	1	1,31	2,78	11,45	20,56	8,48	7,83	38,19
1	0	0	0	1	1	1,00	2,78	9,27	20,23	10,01	7,21	38,45
0	1	0	0	1	1	1,04	4,51	11,57	20,43	10,15	7,29	38,90
0	0	1	0	1	1	1,12	2,78	11,14	20,89	10,45	7,46	39,93
0	0	0	1	1	1	1,09	2,78	9,27	22,89	10,33	7,39	41,70

Table D3 : Results for three enlarged fifths ; sorted for a best value of $q + p_{E4} + p_{A3} + p_{D4}$

B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
1	1	1	1	0	0	0,90	4,27	12,75	16,78	6,52	5,21	29,41
1	1	1	0	1	0	0,80	4,12	12,40	17,40	8,13	5,21	31,55
1	1	1	0	0	1	0,87	4,23	12,66	16,95	7,82	6,96	32,60
1	0	1	1	1	0	0,79	4,09	11,02	18,92	8,09	5,21	33,02
0	1	1	1	1	0	0,84	2,78	10,66	19,17	8,19	5,21	33,41
1	1	0	1	1	0	0,86	4,22	12,63	19,31	8,24	5,21	33,63
1	0	1	1	0	1	0,85	4,20	11,16	18,60	7,79	6,91	34,16
0	1	1	1	0	1	0,91	2,78	10,78	18,85	7,88	7,03	34,67
1	1	0	1	0	1	0,94	4,35	12,93	18,99	7,93	7,10	34,95
1	0	1	0	1	1	0,77	4,06	10,98	19,02	9,20	6,75	35,74
0	1	1	0	1	1	0,81	2,78	10,63	19,27	9,37	6,84	36,30
1	1	0	0	1	1	0,84	4,18	12,54	19,40	9,46	6,89	36,60
0	0	1	1	1	1	0,80	2,78	9,27	20,76	9,31	6,81	37,67
1	0	0	1	1	1	0,82	4,15	11,09	20,94	9,39	6,86	38,01
0	1	0	1	1	1	0,88	2,78	10,73	21,33	9,58	6,96	38,75

Table D4 : Results for four enlarged fifths ; sorted for a best value of $q + p_{E4} + p_{A3} + p_{D4}$

B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
1	1	1	1	1	0	0,67	3,90	11,87	18,33	7,85	5,21	32,07
1	1	1	1	0	1	0,71	3,97	12,05	18,02	7,59	6,64	32,97
1	1	1	0	1	1	0,65	3,87	11,82	18,43	8,81	6,52	34,41
1	0	1	1	1	1	0,64	3,85	10,70	19,65	8,77	6,50	35,56
0	1	1	1	1	1	0,68	2,78	10,39	19,89	8,88	6,56	36,01
1	1	0	1	1	1	0,69	3,94	11,96	20,01	8,94	6,60	36,25

Table D5 : Results for five enlarged fifths ; sorted for a best value of $q + p_{E4} + p_{A3} + p_{D4}$

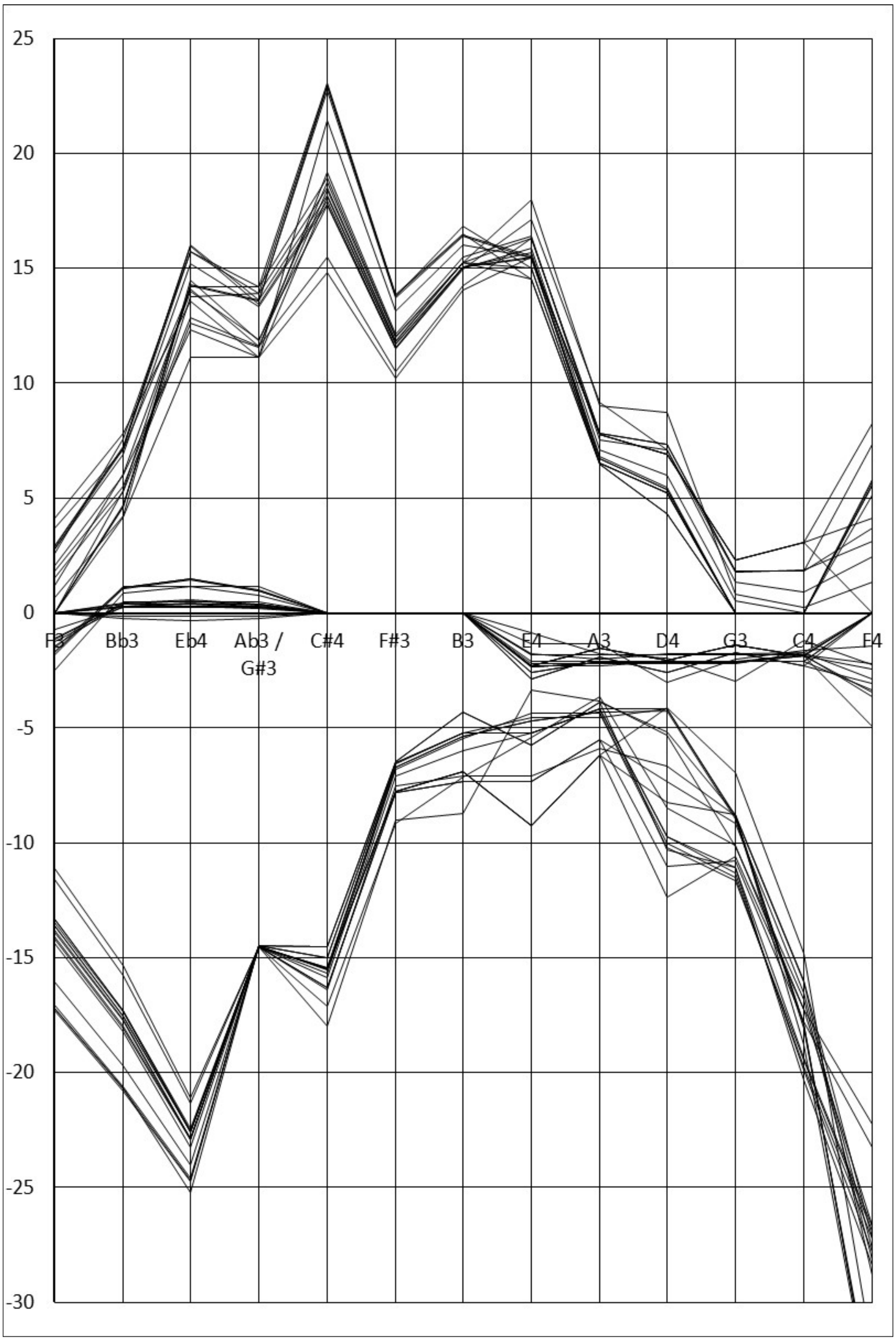
B*b*3	E*b*4	A*b*3	C#4	F#3	B	q	p_{Bb3}	p_{Eb4}	p_{E4}	p_{A3}	p_{D4}	$q + p_{E4} + p_{A3} + p_{D4}$
1	1	1	1	1	1	0,56	3,72	11,45	19,06	8,48	6,34	34,44

Table D6 : Result for six enlarged fifths

Appendix E Tuning table (for A4 = 440 ; ***hence : <u>NOT</u>* <u>415</u> !)**

A3=220	3 thirds et 5 fifths equal		
Partition F3 F4	Batte-ments	tierces majeures	Batte-ments
A4 A3	0.0	F3 A3	1.9
A3 D4	1.9	F3 A3	12.5
D4 G3	-1.9	G3 B3	1.9
G3 C4	1.9	G#3 C4	12.3
C4 F3	-1.2	A3 C#4	8.3
F3 F4	0.0	A3 D4	5.8
F4 A3	0.4	B3 D#4	16.2
A3 D#4	-0.4	C4 E4	1.9
D#4 G#3	0.4	C#4 F4	19.5
G#3 C#4	0.0	F4 A4	3.9
C#4 F3	0.0		
F3 B3	0.0		
B3 E4	1.9		
E4 A3	-1.9		
E4 A4	-3.9		

A3=220	3 thirds and 6 fifths almost equal		
Partition F3 F4	Batte-ments	tierces majeures	Batte-ments
A4 A3	0.0	F3 A3	1.7
A3 D4	1.9	F3 A3	12.9
D4 G3	-1.9	G3 B3	1.9
G3 C4	1.8	G#3 C4	12.2
C4 F3	-1.5	A3 C#4	8.0
F3 F4	0.0	A3 D4	5.6
F4 A3	0.5	B3 D#4	16.4
A3 D#4	-0.5	C4 E4	2.0
D#4 G#3	0.5	C#4 F4	20.4
G#3 C#4	0.0	F4 A4	3.3
C#4 F3	0.0		
F3 B3	0.0		
B3 E4	1.9		
E4 A3	-2.1		
E4 A4	-4.3		


25
20
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5
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F3
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F#3
B3
E4
A3
D4
G3
C4
F4