Urban Economics and Analysis

Part of P. v. Mouche

Assignment A; 2022-2223

Exercise 1 *Consider the production function* $f(k) = k^\alpha$ *where* $\alpha > 0$. *So there is only one production factor* k *and thus the the input also is* k. *The production factor price is* w *(supposed to be positive).*

a. *Show the following:*

$$\begin{aligned} \alpha > 1 &\Leftrightarrow \text{ there is increasing returns to scale;} \\ \alpha = 1 &\Leftrightarrow \text{ there is constant returns to scale;} \\ \alpha < 1 &\Leftrightarrow \text{ there is decreasing returns to scale.} \end{aligned}$$

b. *Determine the conditional production factor demand function* $k^\star(q)$, *i.e. the value of the cost minimising production factor* k *as a function of output* q.

c. *Let* $C(q)$ *be the cost function, i.e. the minimal costs in order to produce an output* q. *Show that* $C(q) = wq^{\frac{1}{\alpha}}$.

d. *Determine the average cost function* $\mathrm{AC}(q) = C(q)/q$ *and prove (the in Slides A mentioned result):*

$$\begin{aligned} \textit{increasing returns to scale} &\Rightarrow \mathrm{AC} \textit{ is decreasing}; \\ \textit{constant returns to scale} &\Rightarrow \mathrm{AC} \textit{ is constant}; \\ \textit{decreasing returns to scale} &\Rightarrow \mathrm{AC} \textit{ is increasing}. \end{aligned}$$

Exercise 2 *An important production function (and also utility funciton) is the Cobb-Douglas function*

$$f(k_1, k_2) = k_1^{\alpha_1} k_2^{\alpha_2};$$

here α_1 *and* α_2 *are positive. In this exercise we eplore its returning to scale properties and ind doing so generalize Exercise 1a. (So here the input is* (k_1, k_2)*).)*

a. *Calculate* $f(\lambda_1 k_1, \lambda k_2)$.

b. Show that

$$\begin{aligned} \textit{increasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 > 1; \\ \textit{constant returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 = 1; \\ \textit{decreasing returns to scale} &\Leftrightarrow \alpha_1 + \alpha_2 < 1. \end{aligned}$$

Exercise 3 *Consider the problem of minimizing total transport costs for transporting over a given distance an input from a mine to a factory and for transporting the output to a market. We assume a 1-dimensional situation. The distance between market and mine is 125 km. The optimal location is the location of the factory where the sum of the transport costs of a given input and the to this input belonging output is minimal.*

Suppose the transport cost function (per ton) for the input is $I(d) = \frac{1}{2}\sqrt{d}$ euro and that for the output (per ton) is $O(d) = \sqrt{d}$ euro; here d is the distance in km.

a. Why these cost functions may be not so realistic?

b. Is this a weight gaining industry or weight losing industry?

c. Determine the total transport costs $T(k_0)$ if the factory is located at distance k_0 from the market.

d. Determine with a calculation the optimal k_0 and also sketch the graph of T.

e. Why, in fact, is it not necessary to determine the optimal k_0 by a calculation?

Exercise 4 *Make exercise 1.2 from the text book.*

Exercise 5 *In this exercise, we analyse the urban model from Chapter 2 in the text book for the residents utility function $u(q, c) = qc$.*

a. Determine the formulas for the optimal floor space $q^\star(x)$ and bread consumption $c^\star(x)$ in terms of the income y, distance to the center x, rental price $p(x)$ and commuting costs t.

b. We know (by the equilibrium principle) that in the equilibrium the utility is independent of x; say this utility is w. Show that for the equilibrium rental price $p^\star(x)$ the equality $(y - tx)^2 = 4wp^\star(x)$ holds.

c. Show that $\frac{dp^\star}{dx}(x) = -\frac{t}{q^\star(x)}$.

d. Show that $p^\star(x) = \frac{p^\star(0)}{y^2}(y - tx)^2$.

e. Can we say something about the exact value of $p^\star(0)$ in part d?

f. Show that total equilibrium rent $p^\star(x) \cdot q^\star(x)$ is a decreasing function of x. Is this realistic?

g. *If You like then now suppose the utility function is the Cobb-Douglas utility function* $u(q,c) = q^{\alpha_1} c^{\alpha_2}$ *with* $\alpha_1 + \alpha_2 = 1$ *and show that the formula in c continues to hold. (In fact this formula holds for a large class of 'nice' utility functions.) Is the total equilibrium rent still a decreasing function of* x*?*

Exercise 6 *Consider the model for the Urban Spatial Structure (Chapter 2, 3 (and 4) in Textbook).* x *denotes the radial distance to the Central Business District. We suppose one income group. Are the following statements false or true?*

a. *Rental price* p *is a decreasing function of* x*.*

b. *Floor space* q *is an increasing function of* x*.*

c. *Amount of bread c is a decreasing function of* x*.*

d. *Price of land* r *is a decreasing function of* x*.*

e. *Building height* h *is a decreasing function of* x*.*

f. *Population density* D *is a decreasing function of* x*.*

Short solutions.

Solution 1 a. $f(\lambda k) = (\lambda k)^\alpha = \lambda^\alpha f(k) \begin{cases} > \lambda f(k) \text{ if } \alpha > 1, \\ = \lambda f(k) \text{ if } \alpha = 1, \\ < \lambda f(k) \text{ if } \alpha < 1. \end{cases}$

b. If the producer wants to produce an amount q with minimal costs, then the input k has to satisfy $k^\alpha = q$. Therefore $k^\star(q) = q^{1/\alpha}$.

c. $C(q) = wk^\star(q)$. Thus, by part b, $C(q) = wq^{\frac{1}{\alpha}}$.

d. For the average costs AC(q) we obtain

$$\mathrm{AC}(q) = \frac{\mathrm{C}(q)}{q} = wq^{\frac{1}{\alpha}-1} = wq^{\frac{1-\alpha}{\alpha}}.$$

e. By the formula in d. For example, for $\alpha = 1/2$ we have decreasing returns to scale and $\mathrm{AC}(q) = wq$. So $AC(q)$ is increasing.

Solution 2

Solution 3 a. There are no fixed costs.

b. Weight gaining industry as $O(d) > I(d)$ (for positive d).

c. $T(k_0) = O(k_0) + I(125 - k_0) = \sqrt{k_0} + \frac{1}{2}\sqrt{125 - k_0}$.

d. Derivative $T'(k_0) = \frac{1}{2\sqrt{k_0}} - \frac{1}{2}\frac{1}{2\sqrt{125-k_0}} = \frac{1}{2}(\frac{1}{\sqrt{k_0}} - \frac{1}{2\sqrt{125-k_0}})$.

We see: derivative zero if $\sqrt{k_0} = 2\sqrt{125 - k_0}$. I.e. $k_0 = 4(125 - k_0)$. We find $k_0 = 100$.

Also derivative is positive if $k_0 < 100$ and negative if $k_0 > 100$. This implies $k_0 = 0$ or $k_0 = 125$ is a minimiser. As $T(125) = \sqrt{125} > \frac{1}{2}\sqrt{125} = T(0)$, 0 is the minimiser.

d. Optimal to locate at the market.

e. Not necessary as we as theory also predecits this result: we have to do with a weight gaining industry and there are economics of distance (as average transportation costs for input and for outputs are decreasing).

Solution 4 a. One factory in city D gives minimises the costs.

b. One factory in each of the four cities minimises the costs.

c. Scale economies are weaker in part b compared to part a.

d. $t = 4$.

Solution 5 a. Remember the general formula for the Marshallian demand functions of the Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} :$$

$$x_1^\star = \frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{m}{p_1}, \quad x_2^\star = \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{m}{p_2}.$$

For our case this leads to

$$q^\star(x) = \frac{y - tx}{2p(x)}, \quad c^\star(x) = \frac{y - tx}{2}.$$

b. We have $u(q^\star(x), c^\star(x)) = w$. So $w = \frac{y-tx}{2p^\star(x)} \cdot \frac{y-tx}{2} = \frac{(y-tx)^2}{4p^\star(x)}$. Thus $(y - tx)^2 = 4wp^\star(x)$.

c. By differentiating the identity in part b with respect to x we find $-2t(y - tx) = 4w\frac{dp^\star}{dx}$. From this and parts a and b we obtain $\frac{dp^\star}{dx} = \frac{-2t(y-tx)}{4w} = \frac{-2t(y-tx)}{4\frac{(y-tx)^2}{4p^\star(x)}} = \frac{-2tp^\star(x)}{y-tx} = \frac{-t}{q^\star(x)}$.

d. Since utility is independent of x we have $w = q^\star(0)c^\star(0) = \frac{y^2}{4p^\star(0)}$. So, by part b, $p^\star(x) = \frac{(y-tx)^2}{4w} = \frac{(y-tx)^2}{4\frac{y^2}{4p^\star(0)}} = \frac{p^\star(0)}{y^2}(y - tx)^2$.

e. No.

f. By part a, we obtain $p^{\star}(x)q^{\star}(x) = \frac{y-tx}{2}$. It is difficult to say whether this is realistic. Remember that we now from theory that $p^{\star}$ is a decreasing function of x and $q^{\star}$ is an increasing function of x. (Also see discussion in text book.)

f. Repeat the above, but start with

$$q(x) = \alpha_1 \frac{y - tx}{p(x)}, \quad c(x) = \alpha_2(y - tx).$$

(Also remember: from theory we know that the formula in c even holds for each 'standard' utility function.)

Solution 6 All are true.