Solution to the “market for lemons problem”
(a) Suppose not. Could demand for used cars equal supply? There are two cases to consider. If $E(\theta|p) < p$, then the conditional average quality of cars on the market is below the price and buyers receive negative expected utility. So there is no demand and the market cannot clear. And if $E(\theta|p) > p$ then buyers receive positive expected utility and all potential buyers would like to buy but not all sellers would like to sell (those with high quality car will not sell as the price is less than average quality). So there is excess demand. The equilibrium outcome is driven by positive selection: a higher price increases the average quality of the cars available on the market. So there are potentially many equilibria solving the pricing condition $E(\theta|p) = p$. There might exist a high price equilibrium where sellers put their high quality cars on the market and buyers are willing to pay the high price. There might also exist a low price equilibrium where the sellers remove the best cars from the market and buyers are only willing to pay a low price.

(b) Fixing the price $p$, if $u, (\theta|p) = p - \frac{\theta}{2}$, all sellers with quality $\theta \leq 2p$ will prefer selling their car to not selling. Thus the average quality of cars conditional on price $p$ is given by $E(\theta|p) = \min(\frac{2p}{2}, \frac{1}{2}) = p$
which is also the competitive equilibrium outcome. Thus any price $p \in [0, \frac{1}{2}]$ is an equilibrium with only cars of quality $\theta \leq 2p$ traded, so that the average quality is just equal to the price.

(c) When $u, (\theta|p) = p - \sqrt{\theta}$, only sellers with quality $\theta \leq p^2$ will prefer to sell. Thus the quality threshold $\theta^* = p^2$ which yields the conditional average quality of cars $E(\theta|p) = \min(\frac{p^2}{2}, \frac{1}{2}) = p$ and $p = 0$ is the only possible solution.

(d) Similar to cases (b) and (c) we have $E(\theta|p) = \min(\frac{p^2}{2}, \frac{1}{2}) = p$. Such that we obtain $p = \sqrt{\frac{1}{8}}$. Another equilibrium is $p = 0$.

(e) Clearly there is market failure because high quality cars do not sell. Pareto improvements are not possible unless we change the game. One possibility is that sellers give guarantees.