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ON A FIXED POINT PROBLEM TRANSFORMATION METHOD

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Abstract. We show how the fixed point problem for a special type of correspondence $\mathbf{R}$ which satisfies a factorisation property can be handled by considering an associated more simple fixed point problem for a correspondence B with domain typically a subset of $\mathbb{R}$. In addition we analyse the fixed point problem for B under additional conditions on $\mathbf{R}$ that guarantee that B is at most singleton-valued. In fact we generalize, improve and make more conceptual a game theoretic technique developed by Selten and Szidarovszky.
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1. Introduction

Given a game in strategic form with n players, its set of Nash equilibria equals the set of fixed points of the best reply correspondence $\mathbf{R}$. In case each strategy set is m-dimensional, this fixed point problem is a mn-dimensional one.

For oligopoly-like games, [9] and [12] independently where able to transform the fixed point problem for $\mathbf{R}$ into an associated more simple fixed point problem for a correspondence B with domain in $\mathbb{R}$. This technique was used by various authors dealing with such games.[1] One aim of our article is to improve this technique and to make it more conceptual and general. To this

[1]For example by [11, 5, 2, 10, 6, 7, 13, 1, 14, 8].

end we divide it into two parts: 1) the definition of B and the relation between the two fixed point problems; 2) the analysis of the fixed point problem for B. Both parts will be performed in a correspondence setting where games do not play a role. Later on the results will be applied for games in strategic form to the best reply correspondence.

Theorem 3.2 identifies a quite general setting for which the first part continues to work. This setting is compatible with games in strategic form where for each player i his best reply correspondence depends on some weighted sum $\sum_l \varphi_l(x_l)$ (with values in an Abelian group G) of the strategies x_l of the other players; we refer to this correspondence as the *reduced best reply correspondence* $\tilde{R}_i$. In particular the setting contains all additively aggregative games, i.e., games where the payoff function of a player depends only on his own strategy and the sum of all strategies.

Another aim of our article is to study the fixed point problem for B (in case $G = \mathbb{R}$), though under additional assumptions which enable us to prove the existence of a fixed point of B by means of the intermediate value theorem. Applying our results to games, we obtain a Nash equilibrium existence result which has been proved in the literature only by means of the Nikaido-Isoda theorem ([4]) or related theorems which rely on Brouwer's fixed point theorem. The main assumption made is that each correspondence R_i is singleton-valued and that each function $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is continuous and strictly increasing. Among other things these assumptions entail the at-most-singleton-valuedness of B and simplifies the analysis. We also show that under the additional condition that every $\varphi_i \circ \tilde{R}_i$ is decreasing there exists a unique Nash equilibrium. As far as we know this uniqueness result is new. It should be noted that in particular no differentiability assumption is made throughout the whole article.

2. Setting

Let n be a positive integer,[2]

$$N \;:=\; \{1, \ldots, n\}, \tag{2.1}$$

and let

$$\varphi_i : X_i \to G \ (i \in N)$$

be mappings from a non-empty set X_i into an Abelian group G.

Put

$$\mathbf{X} \;:=\; X_1 \times \cdots \times X_n \tag{2.2}$$

and for $i \in N$

$$\mathbf{X}_{\hat{\imath}} \;:=\; X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n. \tag{2.3}$$

We sometimes identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$.

[2]Our results formally hold for $n = 1$ when empty objects are interpreted correctly.

For $i \in N$ let $R_i \colon \mathbf{X}_{\hat{\imath}} \multimap X_i$ be a correspondence. The correspondence $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ is defined by

$$\mathbf{R}(\mathbf{x}) := R_1(\mathbf{x}_{\hat{1}}) \times \cdots \times R_n(\mathbf{x}_{\hat{n}}). \tag{2.4}$$

We write, for $i \in N$,[3]

$$T_i := \sum_{l \in N \setminus \{i\}} \varphi_l(X_l). \tag{2.5}$$

Define the mapping $\varphi : \mathbf{X} \to G$ by

$$\varphi(\mathbf{x}) := \sum_{l \in N} \varphi_l(x_l)$$

and let

$$Y := \varphi(\mathbf{X}).$$

Note that for every $i \in N$

$$Y = T_i + \varphi_i(X_i).$$

For $i \in N$, we say that R_i has the *factorisation property* if there exists[4] a correspondence $\tilde{R}_i : T_i \multimap X_i$ such that for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$

$$R_i(\mathbf{z}) = \tilde{R}_i\Big(\sum_{l \in N \setminus \{i\}} \varphi_l(z_l) \Big). \tag{2.6}$$

Note that R_i is at most singleton-valued if and only if $\tilde{R}_i$ is at most singleton-valued. Finally, let $\mathcal{Y}$ be a subset of Y such that for all $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$ one has $\varphi(\mathbf{x}) \in \mathcal{Y}$. Thus, for instance, $\mathcal{Y} = Y$ is always possible.

3. Transformation method

In the following definition we introduce our most important objects: B_i ($i \in N$) and B. This definition was inspired by the articles like the ones mentioned in the Introduction. The B_i are quite similar to what in [13, p. 42] are called the *cumulative best reply correspondences*.

Definition 3.1. Let $i \in N$ and suppose R_i has the factorisation property. The correspondence $B_i \colon \mathcal{Y} \multimap X_i$ is defined by

$$B_i(y) := \{ x_i \in X_i \mid y - \varphi_i(x_i) \in T_i \text{ and } x_i \in \tilde{R}_i(y - \varphi_i(x_i)) \}.$$

If every R_i has the factorisation property, then the B_i ($i \in N$) are well-defined and hence we are in a position to define the correspondences $\mathbf{B} : \mathcal{Y} \multimap \mathbf{X}$ and $B : \mathcal{Y} \multimap G$ by

$$\mathbf{B}(y) := B_1(y) \times \cdots \times B_n(y), \quad B(y) := \varphi(\mathbf{B}(y)). \ \diamond$$

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[3]The sum here is a Minkowski-sum.

[4]If such $\tilde{R}_i$ exists, then it is unique.

Theorem 3.2. *Suppose every R_i has the factorisation property.*

(1) $\varphi(\mathrm{fix}(\mathbf{R})) = \mathrm{fix}(B)$. *So* $\mathbf{R}$ *has a fixed point if and only if* B *has a fixed point.*
(2) $\mathrm{fix}(\mathbf{R}) \subseteq \mathbf{B}(\mathrm{fix}(B))$.
(3) *Let* $y \in \mathcal{Y}$. *If* $\mathbf{x} \in \mathbf{B}(y)$ *and* $B(y) = \{y\}$, *then* $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$.
(4) *If* B *is at most singleton-valued on* $\mathrm{fix}(B)$, *then* $\mathrm{fix}(\mathbf{R}) = \mathbf{B}(\mathrm{fix}(B))$.
(5) *If* B *is at most singleton-valued and has a unique fixed point, then* $\mathbf{R}$ *has a unique fixed point.* $\diamond$

Proof. 1. '$\supseteq$': suppose $y \in \mathrm{fix}(B)$. So $y \in B(y) = \varphi(\mathbf{B}(y)) = \sum_{l \in N} \varphi_l(B_l(y))$. Let $x_i \in B_i(y)$ $(i \in N)$ be such that $y = \sum_{l \in N} \varphi_l(x_l)$. So $y = \varphi(\mathbf{x})$. This implies $x_i \in \tilde{R}_i(y - \varphi_i(x_i)) = R_i(\mathbf{x}_{\hat{\imath}})$ $(i \in N)$. Thus $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$ and $y \in \varphi(\mathrm{fix}(\mathbf{R}))$.

'$\subseteq$': suppose $y \in \varphi(\mathrm{fix}(\mathbf{R}))$. Let $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$ be such that $y = \varphi(\mathbf{x})$. As $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$ we have for every $i \in N$ that $x_i \in R_i(\mathbf{x}_{\hat{\imath}}) = \tilde{R}_i(\varphi(\mathbf{x}) - \varphi_i(x_i)) = \tilde{R}_i(y - \varphi_i(x_i))$. It follows that $x_i \in B_i(y)$ $(i \in N)$. Now $y = \sum_{i \in N} \varphi_i(x_i) \in \sum_{i \in N} \varphi_i(B_i(y)) = B(y)$. Thus $y \in \mathrm{fix}(B)$.

2. Suppose $\mathbf{e} \in \mathrm{fix}(\mathbf{R})$. Note that $y := \varphi(\mathbf{e}) \in \mathcal{Y}$. We prove that $y \in \mathrm{fix}(B)$ and $\mathbf{e} \in \mathbf{B}(y)$. For $i \in N$ we have $y - \varphi_i(e_i) \in T_i$. As $\mathbf{e} \in \mathrm{fix}(\mathbf{R})$, we have for every $i \in N$ that $e_i \in R_i(\mathbf{e}_{\hat{\imath}}) = \tilde{R}_i(y - \varphi_i(e_i))$. So $e_i \in B_i(y)$ and hence $\mathbf{e} \in \mathbf{B}(y)$. From this, $y = \sum_{l \in N} \varphi_l(e_l) \in \sum_{l \in N} \varphi_l(B_l(y)) = B(y)$. Thus $y \in \mathrm{fix}(B)$.

3. Fix $i \in N$. Note that $\varphi(\mathbf{x}) = \sum_{l \in N} \varphi_l(x_l) \in \sum_{l \in N} \varphi_l(B_l(y)) = B(y)$. As $B(y) = \{y\}$, $\varphi(\mathbf{x}) = y$ follows. As $x_i \in B_i(y)$, we obtain $x_i \in \tilde{R}_i(y - \varphi_i(x_i))$, as desired.

4. As part 2 holds, we still have to prove '$\supseteq$'. So suppose $\mathbf{x} \in \mathbf{B}(\mathrm{fix}(B))$. Let $y \in \mathrm{fix}(B)$ such that $\mathbf{x} \in \mathbf{B}(y)$. As $y \in B(y)$ and B is at most singleton-valued on $\mathrm{fix}(B)$, we have $B(y) = \{y\}$. So by part 3, $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$.

5. From parts 1 and 4. $\square$

4. Analysis of B

The analysis in this section is divided into two steps. First we analyse the B_i $(i \in N)$ in Proposition 4.1 and then in Proposition 4.2 we analyse B. ¿From now on we shall always assume $\mathcal{Y} = Y$.

Proposition 4.1. *Fix* $i \in N$. *Suppose* $G = \mathbb{R}$ *and* $T_i \subseteq \mathbb{R}_+$. *Suppose* R_i *has the factorisation property and is singleton-valued.*[5] *Consider the correspondence* $B_i : Y \multimap X_i$.

(1) *Let* $y \in Y$ *and suppose* $x_i \in B_i(y)$. *Then* $\varphi_i(x_i) \leq y$.

[5] So also $\tilde{R}_i : T_i \multimap X_i$ is singleton-valued and henceforth we consider $\tilde{R}_i$ as a function $T_i \to X_i$.

(2) For all $z \in T_i$: $\tilde{R}_i(z) \in B_i((\varphi_i \circ \tilde{R}_i)(z) + z)$. In particular $\tilde{R}_i(0) \in B_i((\varphi_i \circ \tilde{R}_i)(0))$.

(3) For every $y \in Y$: $B_i(y) \neq \emptyset \Leftrightarrow y \in (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$.

(4) If the function $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is injective, then $\varphi_i \circ B_i$ is at most singleton-valued and is singleton-valued on $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$.

(5) (a) If the function $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is increasing, then for all $y, y' \in Y$ with $y < y'$ and $x \in B_i(y), x' \in B_i(y')$ one has $\varphi_i(x') - \varphi_i(x) < y' - y$.

(b) If the function $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is strictly increasing, then $\varphi_i \circ B_i - \mathrm{Id}$ is strictly decreasing as a function on $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$

(6) If $\varphi_i \circ \tilde{R}_i$ is decreasing and $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is strictly increasing, then for every $y, y' \in (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ with $y < y'$ it holds that[6] $\varphi_i(B_i(y)) \geq \varphi_i(B_i(y'))$. ⋄

Proof. Note that $B_i(y) = \{x_i \in X_i \mid \varphi_i(x_i) \in y - T_i \text{ and } x_i \in \tilde{R}_i(y - \varphi_i(x_i))\}$.

1. As $T_i \subseteq \mathbb{R}_+$, one has $y - T_i \leq y$.

2. $B_i((\varphi_i \circ \tilde{R}_i)(z) + z) = \{x_i \in X_i \mid \varphi_i(x_i) \in (\varphi_i \circ \tilde{R}_i)(z) + z - T_i \text{ and } x_i = \tilde{R}_i((\varphi_i \circ \tilde{R}_i)(z) + z - \varphi_i(x_i))\}$. Thus $\tilde{R}_i(z) \in B_i((\varphi_i \circ \tilde{R}_i)(z) + z)$.

3. '⇒': suppose $x_i \in B_i(y)$. Now $x_i = \tilde{R}_i(y - \varphi_i(x_i))$ and $y - \varphi_i(x_i) \in T_i$. This implies $y = \varphi_i(x_i) + (y - \varphi_i(x_i)) = (\varphi_i \circ \tilde{R}_i)(y - \varphi_i(x_i)) + (y - \varphi_i(x_i)) \in (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$.

'⇐': by part 1.

4. Suppose $y_0 \in Y$ and $y, y' \in (\varphi_i \circ B_i)(y_0)$. Let $x, x' \in B_i(y_0)$ be such that $y = \varphi_i(x)$ and $y' = \varphi_i(x')$. As $\tilde{R}_i$ is singleton-valued it follows that $x = \tilde{R}_i(y_0 - \varphi_i(x))$ and $x' = \tilde{R}_i(y_0 - \varphi_i(x'))$. Thus

$$y = (\varphi_i \circ \tilde{R}_i)(y_0 - \varphi_i(x)) + (y_0 - \varphi_i(x)) = (\varphi_i \circ \tilde{R}_i)(y_0 - \varphi_i(x')) + (y_0 - \varphi_i(x')).$$

As $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is injective, it follows that $y_0 - \varphi_i(x) = y_0 - \varphi_i(x')$. So $\varphi_i(x) = \varphi_i(x')$. Thus $y = y'$ and the first statement holds. Part 3 now implies the second statement.

5a. By contradiction suppose $y, y' \in Y$ with $y < y'$, $x \in B_i(y), x' \in B_i(y')$ and $\varphi_i(x') - \varphi_i(x) \geq y' - y$. Now $y' - \varphi_i(x') \leq y - \varphi_i(x)$ and

$$\begin{gathered}\varphi_i(x') - \varphi_i(x) = (\varphi_i \circ \tilde{R}_i)(y' - \varphi_i(x')) - (\varphi_i \circ \tilde{R}_i)(y - \varphi_i(x)) \\ = \Big(((\varphi_i \circ \tilde{R}_i)(y' - \varphi_i(x')) + (y' - \varphi_i(x')) - ((\varphi_i \circ \tilde{R}_i)(y - \varphi_i(x)) + (y - \varphi_i(x))\Big) \\ +(\varphi_i(x') - \varphi_i(x)) + (y - y') < 0 + \varphi_i(x') - \varphi_i(x) + 0,\end{gathered}$$

which is absurd.

5b. This follows from parts 3, 4 and 5a.

6. From parts 2 and 3 it follows that B_i is singleton-valued on

$$(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i).$$

[6]Note that, by parts 5 and 6, the sets $B_i(y)$ and $B_i(y')$ are singletons.

7. By contradiction suppose $y, y' \in (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ with $y < y'$ and $\varphi_i(B_i(y)) < \varphi_i(B_i(y'))$. Now $(\varphi_i \circ \tilde{R}_i)(y - \varphi_i(B_i(y))) = \varphi_i(B_i(y)) < \varphi_i(B_i(y')) = (\varphi_i \circ \tilde{R}_i)(y' - \varphi_i(B_i(y')))$. As $\varphi_i \circ \tilde{R}_i$ is decreasing, we have $y - \varphi_i(B_i(y)) > y' - \varphi_i(B_i(y'))$. As $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is strictly increasing it follows that $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(y - \varphi_i(B_i(y))) > (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(y' - \varphi_i(B_i(y')))$. Thus $y > y'$, which is a contradiction. □

In Proposition 4.2 and in Theorem 4.3 we assume that every R_i is singleton-valued and has the factorisation property. Also we assume that $G = \mathbb{R}$ and for every $i \in N$ that $\varphi_i \geq 0$ and $T_i = [0, \mu_i]$ with $\mu_i > 0$ or $T_i = \mathbb{R}_+$. This implies that $n \geq 2$, that $0 \in \varphi_i(X_i)$ $(i \in N)$ and that $\varphi_i(X_i) \subseteq T_j$ for every $i, j \in N$ with $i \neq j$. We put

$$N_T := \{i \in N \mid T_i = [0, \mu_i]\}$$

and fix $l \in N$ such that $(\varphi_i \circ \tilde{R}_i)(0) \leq (\varphi_l \circ \tilde{R}_l)(0)$ $(i \in N)$. Finally,

$$\underline{X} := (\varphi_l \circ \tilde{R}_l)(0).$$

Proposition 4.2. *Suppose every $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ is an interval of $\mathbb{R}$ with for $i \in N_T$ a well-defined maximum w_i. If $N_T \neq \emptyset$, then fix $k \in N_T$ such that $w_k \leq w_i$ $(i \in N_T)$. Let $I := [\underline{X}, +\infty[$ if $N_T = \emptyset$ and $I := [\underline{X}, w_k]$ if $N_T \neq \emptyset$. Consider the correspondences B_i $(i \in N)$ and B. Finally, suppose each correspondence $\varphi_i \circ B_i$ is at most singleton-valued.*

(1) I is a non-empty interval.
(2) $I \subseteq \cap_{i \in N} (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$.
(3) $B \upharpoonright I$ is singleton-valued.
(4) If $\varphi_l \circ \tilde{R}_l + \mathrm{Id}$ is increasing, then $B(y) = \emptyset$ for every $y \in Y \setminus I$.[7]
(5) $B(\underline{X}) \geq \underline{X}$.
(6) For every $i \in N \setminus N_T$ suppose that the function $\varphi_i \circ \tilde{R}_i$ is bounded. If $N_T \neq \emptyset$, then suppose $\varphi_k \circ \tilde{R}_k + \mathrm{Id}$ is increasing.
  (a) There exists $\overline{X} \in I$ with $B(\overline{X}) \leq \overline{X}$;
  (b) If the function $B \upharpoonright I$ is continuous, then this function has a fixed point. ⋄

Proof. 1. This is trivial if $N_T = \emptyset$. Now suppose $N_T \neq \emptyset$. We prove that the inequality $(\varphi_l \circ \tilde{R}_l)(0) \leq w_k$ holds.

Case where $l = k$: here it holds as $(\varphi_l \circ \tilde{R}_l)(0) \in (\varphi_l \circ \tilde{R}_l + \mathrm{Id})(T_l)$.

Case where $l \neq k$: as $(\varphi_l \circ \tilde{R}_l)(0) \in \varphi_l(X_l) \subseteq T_k$ it follows that

$$\begin{aligned}(\varphi_l \circ \tilde{R}_l)(0) &\leq (\varphi_l \circ \tilde{R}_l)(0) + (\varphi_k \circ \tilde{R}_k)(\varphi_l \circ \tilde{R}_l)(0)) \\ &\leq \max(\varphi_k \tilde{R}_k + \mathrm{Id})(T_k) = w_k.\end{aligned}$$

[7]And by part 3 we see that B is almost singleton-valued.

2. Case $N_T = \emptyset$: now $T_i = \mathbb{R}_+$ $(i \in N)$. As $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(\mathbb{R})$ is an interval and $\varphi_i \geq 0$, it follows that this interval contains $[(\varphi_i \circ \tilde{R}_i)(0), +\infty\,[$. So $\cap_{i \in N}((\varphi_i \circ \tilde{R}_i + \mathrm{Id})(\mathbb{R}))$ contains $\cap_{i \in N}[(\varphi_i \circ \tilde{R}_i)(0), +\infty\,[\, = [\underline{X}, +\infty\,[\, = I$.

Case $N_T \neq \emptyset$: for $i \in N_T$, as $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ is an interval, it follows that this interval contains $[(\varphi_i \circ \tilde{R}_i)(0), w_i]$. This implies $I = [\underline{X}, w_k] = [(\varphi_l \circ \tilde{R}_l)(0), w_k] \subseteq \cap_{i \in N_T}[(\varphi_i \circ \tilde{R}_i)(0), w_i] \subseteq \cap_{i \in N_T}(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$. Also $[(\varphi_l \circ \tilde{R}_l)(0), w_k] \subseteq \cap_{i \in N \setminus N_T}[(\varphi_i \circ \tilde{R}_i)(0), w_k] \subseteq \cap_{i \in N \setminus N_T}[(\varphi_i \circ \tilde{R}_i)(0), +\infty\,[\, \subseteq \cap_{i \in N \setminus N_T}(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$. Thus $I \subseteq \cap_{i \in N}(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$.

3. With Proposition 4.1(3) it follows that every $\varphi_i \circ B_i$ is singleton-valued on $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$. So every $\varphi_i \circ B_i$ is singleton-valued on $\cap_{j \in N}(\varphi_j \circ \tilde{R}_j + \mathrm{Id})(T_j)$. Part 2 now implies that every $\varphi_i \circ B_i$ is singleton-valued on I. Thus also $B \restriction I$ is singleton-valued.

4. As $\varphi_l \circ \tilde{R}_l + \mathrm{Id}$ is increasing and $0 = \min(T_l)$, we have $\varphi_l \circ \tilde{R}_l + \mathrm{Id} \geq (\varphi_l \circ \tilde{R}_l + \mathrm{Id})(0)$. Therefore Proposition 4.1(3) implies $B_l(y) = \emptyset$ for every $y \in Y$ with $y < (\varphi_l \circ \tilde{R}_l)(0)$. This implies that $B(y) = \emptyset$ for every $y \in Y$ with $y < \underline{X}$. So if $N_T = \emptyset$, then the proof is complete. Now suppose $N_T \neq \emptyset$. If $y \in Y$ with $y > w_k$, then, by Proposition 4.1(3), $B_k(y) = \emptyset$ and therefore also $B(y) = \emptyset$.

5. With Proposition 4.1(2) we obtain $B(\underline{X}) = \sum_{j \in N}(\varphi_j \circ B_j)(\underline{X})) \geq (\varphi_l \circ B_l)(\underline{X}) = (\varphi_l \circ B_l)((\varphi_l \circ \tilde{R}_l)(0)) = (\varphi_l \circ \tilde{R}_l)(0) = \underline{X}$.

6a. Case $N_T = \emptyset$: we prove that for every $i \in N$ there exists $\overline{X}_i \geq \underline{X}$ such that $(\varphi_i \circ B_i)(y) \leq \frac{y}{n}$ for all $y \geq \overline{X}_i$. (Then take, e.g., $\overline{X} := \max\{\overline{X}_i \mid i \in N\}$.) So fix $i \in N$. By contradiction suppose there does not exist such an $\overline{X}_i$. This implies the existence of a sequence (y_j) in $[\underline{X}, +\infty\,[$ with $\lim_{j \to \infty} y_j = +\infty$ and $(\varphi_i \circ B_i)(y_j) > y_j/n$ for all j. Now for all j

$$\varphi_i(\tilde{R}_i(y_j - (\varphi_i \circ B_i)(y_j))) = \varphi_i(B_i(y_j)) > y_j/n.$$

It follows that $\lim_{j \to \infty} \tilde{R}_i(y_j - (\varphi_i \circ B_i)(y_j)) = +\infty$. As φ_i is bounded, this is absurd.

Case $N_T \neq \emptyset$: take $\overline{X} := (\varphi_k \circ \tilde{R}_k + \mathrm{Id})(\mu_k)$. As $\varphi_k \circ \tilde{R}_K + \mathrm{Id}$ is increasing and in case $k \neq l$ it holds that $\overline{X} \geq \mu_k \geq (\varphi_l \circ \tilde{R}_l)(0)$, it follows that $\overline{X} \in [(\varphi_l \circ \tilde{R}_l)(0), w_k]$. From Proposition 4.1(2) it follows that

$$\begin{aligned} B(\overline{X}) = \sum_{j \in N}(\varphi_j \circ B_j)(\overline{X}) &= (\varphi_k \circ B_k)((\varphi_k \circ \tilde{R}_k)(\mu_k) + \mu_k) \\ &+ \sum_{j \neq k}(\varphi_j \circ B_j)((\varphi_k \circ \tilde{R}_k)(\mu_k) + \mu_k) \end{aligned}$$

$$= (\varphi_k \circ \tilde{R}_k)(\mu_k) + \sum_{j \neq k}(\varphi_j \circ B_j)((\varphi_k \circ \tilde{R}_k)(\mu_k) + \mu_k) \leq (\varphi_k \circ \tilde{R}_k)(\mu_k) + \mu_k = \overline{X}.$$

6b. A consequence of parts 5, 6a and the intermediate value theorem. □

Theorem 4.3. *Suppose every R_i is singleton-valued and has the factorisation property, $G = \mathbb{R}$ and for every $i \in N$ that $\varphi_i \geq 0$ and $T_i = [0, \mu_i]$ with $\mu_i > 0$ or $T_i = \mathbb{R}_+$. Also suppose every $\varphi_i \circ \tilde{R}_i$ is bounded and every $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is continuous and strictly increasing. Consider the correspondence B.*

(1) B is at most singleton-valued.
(2) The correspondence B has a fixed point.
(3) If every $\varphi_i \circ \tilde{R}_i$ is decreasing, then B has a unique fixed point.
(4) If for every $i \in N$ and $z \in T_i$ with $\varphi_i(\tilde{R}_i(z)) \leq z$ it holds that $\varphi_i \circ \tilde{R}_i$ is decreasing on $T_i \setminus [0, z]$, then B has a unique fixed point.[8] $\diamond$

Proof. Note that every $\varphi_i \circ \tilde{R}_i$ also is continuous. Every $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ is an interval of $\mathbb{R}$. As $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is strictly increasing, $\varphi_i \circ B_i$ is at most singleton-valued and $(\varphi_i \circ \tilde{R}_i + \mathrm{Id})(T_i)$ has in case $i \in N_T$ as maximum $w_i = (\varphi_i \circ \tilde{R}_i + \mathrm{Id})(\mu_i)$. Let I be as in Proposition 4.2.

1. Apply Proposition 4.2(3,4).

2. Proposition 4.1(5a) together with Proposition 4.2((2) guarantees that the function $B \upharpoonright I$ is continuous. Proposition 4.2(6b) applies and implies that this function has a fixed point. Thus also B has a fixed point.

3. As in the proof of part 1 we see that $B \upharpoonright I$ has a fixed point. We shall prove that this fixed point is unique. Then the proof is complete as by Proposition 4.2(4) we see that also B has a unique fixed point. Well, Proposition 4.1(6) implies that every $\varphi_i \circ B_i$ is decreasing on I. This implies that also $B \upharpoonright I$ is decreasing and so this function has a unique fixed point.

4. As in part 3, the proof is complete if we can prove that $B \upharpoonright I$ has a unique fixed point. By contradiction suppose that y, y' with $y < y'$ are fixed points of $B \upharpoonright I$. So we have $y = B(y) = \sum_l \varphi_l(B_l(y))$. This implies that there exists $M \subseteq N$ with $\#M = n - 1$ such that $\varphi_m(B_m(y)) \leq y/2$ $(m \in M)$. Therefore $(\tilde{R}_m - \mathrm{Id})(y - \varphi_m(B_m(y))) = 2\varphi_m(B_m(y)) - y \leq 0$. Hence, for every $m \in M$,

$$\varphi_m \circ \tilde{R}_m \text{ is decreasing on } T_m \setminus [0, y - \varphi_m(B_m(y))].$$

This, together with Proposition 4.1(5a), implies that for every $m \in M$

$$\varphi_m \circ B_m \text{ is decreasing on } Y \setminus [0, y].$$

Let $\{i\} = N \setminus M$. By Proposition 4.1(5b),

$$\varphi_i \circ B_i - \mathrm{Id} \text{ is strictly decreasing.}$$

It follows that $B - \mathrm{Id}$ is strictly decreasing on $Y \setminus [0, y]$. In the proof of part 1, we have seen that $(B - \mathrm{Id}) \upharpoonright I$ is continuous; this implies that $B - \mathrm{Id}$ is also strictly decreasing on $Y \setminus [0, y\,[$. So $B(y') - y' < B(y) - y = 0$, a contradiction with $B(y') = y'$. □

[8]Note that this result implies part 3 of the theorem.

The above analysis of B becomes much more complicated in case the $\tilde{R}_i$ are multi-valued. For such a situation we now give sufficient conditions for the existence of a fixed point of B by using a deep fixed point result of [3]:

Theorem 4.4. *Suppose every R_i has the factorisation property, every X_i is a non-empty compact subset of $\mathbb{R}$, $G = \mathbb{R}$ and $\varphi_i = \mathrm{Id}$ ($i \in N$). If each correspondence $\varphi_i \circ \tilde{R}_i$ is upper hemi-continuous and has a decreasing single-valued selection, then B has a fixed point.* $\diamond$

Proof. To this situation the result in [3] applies and guarantees that $\mathbf{R}$ has a fixed point. Theorem 3.2 implies that B has a fixed point. $\square$

5. A Nash equilibrium existence and uniqueness result

In this section we apply Theorems 3.2 and 4.3 to a special class of games in strategic form.

We recall that a *game in strategic form* between n (≥ 1) players is given by nonempty (*strategy*) sets X_i ($1 \leq i \leq n$) and (*payoff*) functions $f_i : X_1 \times \cdots \times X_n \to \mathbb{R}$ ($1 \leq i \leq n$).

Consider a game in strategic form Γ. Using notations (2.1), (2.2), (2.3), we define for each *player* i (i.e., for every $i \in N$) and for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ the (*conditional payoff*) function $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ by

$$f_i^{(\mathbf{z})}(x_i) \;:=\; f_i(x_i; \mathbf{z})$$

and the (*best reply*) correspondence $R_i \colon \mathbf{X}_{\hat{\imath}} \multimap X_i$ by

$$R_i(\mathbf{z}) \;:=\; \operatorname{argmax} f_i^{(\mathbf{z})}.$$

The correspondence $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ is defined by (2.4). A *Nash-equilibrium* of Γ is a fixed point of $\mathbf{R}$. Note that $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$, if and only if for all $i \in N$, x_i is a maximiser of $f_i^{(\mathbf{x}_{\hat{\imath}})}$.

Corollary 5.1. *Consider a game in strategic form where the following conditions hold.*

a. Every best-reply correspondence R_i is singleton-valued.

b. There exist $\varphi_i : X_i \to \mathbb{R}$ ($i \in N$) and[9] *$\tilde{f}_i^{(z)} : X_i \to \mathbb{R}$ ($i \in N, z \in T_i$) such that $f_i^{(\mathbf{z})} = \tilde{f}_i^{(\sum_l \varphi_l(z_l))}$ ($i \in N$, $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$).*

c. For every $i \in N$: $\varphi_i \geq 0$, $T_i = [0, \mu_i]$ with $\mu_i > 0$ or $T_i = \mathbb{R}_+$.[10]

[9] Using notation (2.5).

[10] For example, this condition is satisfied if for every X_i is a non-negative orthant and φ_i is a linear function strictly increasing in all variables. (Of course also compact X_i can be considered.)

Noting that, for any $i \in N$, the function $\tilde{R}_i : T_i \to \mathbb{R}$ is well-defined by

$$\tilde{R}_i(z) \;:=\; \text{argmax}\; \tilde{f}_i^{(z)},$$

further assume the following conditions hold.

d. *Each function $\varphi_i \circ \tilde{R}_i + \mathrm{Id}$ is continuous and strictly increasing.*
e. *Each function $\varphi_i \circ \tilde{R}_i$ is bounded.*

Then:

(1) *There exists a Nash equilibrium.*
(2) *If every $\varphi_i \circ \tilde{R}_i$ is decreasing, then the game has a unique Nash equilibrium.*
(3) *If for every $i \in N$ and $z \in T_i$ with $\varphi_i(\tilde{R}_i(z)) \leq z$ it holds that $\varphi_i \circ \tilde{R}_i$ is decreasing on $T_i \setminus [0, z]$, then the game has a unique Nash equilibrium.* $\diamond$

Proof. Note that, with $G = \mathbb{R}$, (2.6) holds, i.e. that every R_i has the factorisation property. Consider the correspondences B_i ($i \in N$) and B.

1. Theorem 4.3(1) guarantees that B has a fixed point. Theorem 3.2(1) guarantees that the game has a Nash equilibrium.

2. Theorem 4.3(2) guarantees that B has a unique fixed point. Theorem 4.3(1) guarantees that B is at most singleton-valued. Hence Theorem 3.2(5) guarantees that the game has a unique Nash equilibrium.

3. Exactly the same proof as in part 2 by replacing there 'Theorem 4.3(2)' by 'Theorem 4.3(4)'. □

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