Exercises - SOLUTIONS
UEC-51806 Advanced Microeconomics,
Instructor: Dusan Drabik, de Leeuwenborch 2105

1. A consumer has a preference relation on $\mathbf{R}^1_+$ which can be represented by the utility function $u(x) = x^2 + 4x + 4$. Is this function quasi-concave? Briefly explain. Is there a concave utility function representing the consumer's preferences? If so, display one; if not, why not?

Solution

Without loss of generality, consider any two points $x_1, x_2 \in \mathbf{R}^1_+$, such that $x_1 < x_2$. Because $du/dx \geq 0$ for $x_1 \in \mathbf{R}^1_+$, we have $u(x_1) < u(x_2)$, from which $u(x_1) = \min\{u(x_1), u(x_2)\}$. Now, form $x^t = tx_1 + (1-t)x_2$ for $t \in [0,1]$. Because $x^t > x_1$ and $du/dx \geq 0$, it must be the case that $u(x^t) \geq u(x_1) = \min\{u(x_1), u(x_2)\}$, hence u(x) is a quasi-concave function.

Alternatively, the set $\{x \geq 0 : u(x) \geq k\} = \{x \geq 0 : x^2 + 4x + 4 \geq k\} = \begin{Bmatrix} \left[\sqrt{k} - 2, \infty\right) \text{ for } k > 4 \\ [0, \infty) \text{ for } k \leq 4 \end{Bmatrix}$ is a convex set for all $k \in \mathbf{R}$.

Yes, there is such a concave utility function, for example: $v(x) = x$, or $v(x) = x^{\frac{1}{2}}$.

2. A consumer has Lexicographic preferences on $\mathbf{R}^2_+$ if the relationship $\succsim$ satisfies $\mathbf{x}^1 \succsim \mathbf{x}^2$ whenever $x_1^1 > x_1^2$, or $x_1^1 = x_1^2$ and $x_2^1 \geq x_2^2$. Show that lexicographic preferences on $\mathbf{R}^2_+$ are rational, i.e., complete and transitive.

Solution

We need to show that Lexicographic preferences on $\mathbb{R}^2_+$ are complete and transitive.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, $x \succsim y \Leftrightarrow x_1 > y_1$ or $[x_1 = y_1$ and $x_2 \geq y_2]$.

Completeness: We need to show that for any $x, y \in \mathbb{R}^2_+$, either $x \succsim y$, $y \succsim x$ or both. For any $x, y \in \mathbb{R}^2_+$, we have exactly one of the following three cases:

1. $x_1 \neq y_1$. Then $x \succsim y$ if $x_1 > y_1$; and $y \succsim x$ if $y_1 > x_1$;
2. $x_1 = y_1$ and $x_2 \neq y_2$. Then $x \succsim y$ if $x_2 > y_2$; and $y \succsim x$ if $y_2 > x_2$;
3. $x = y$ (i.e. $x_1 = y_1$ and $x_2 = y_2$). In this case we have $x \succsim y$ and $y \succsim x$.

Therefore, $\succsim$ is complete.

Transitivity: We need to show that, for any $x, y, z \in \mathbb{R}^2_+$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

$x \succsim y$ implies $x_1 > y_1$ or $[x_1 = y_1$ and $x_2 \geq y_2]$. $y \succsim z$ implies $y_1 > z_1$ or $[y_1 = z_1$ and $y_2 \geq z_2]$.

1. If $x_1 > y_1$, then we have $x_1 > y_1 \geq z_1$. So $x \succsim z$.
2. If $x_1 = y_1$, then we know that $x_2 \geq y_2$. If $y_1 > z_1$, then we have $x_1 = y_1 > z_1$. So $x \succsim z$. If $y_1 = z_1$, then we have $x_1 = y_1 = z_1$ and $x_2 \geq y_2 \geq z_2$. So $x \succsim z$.

3. A consumer with convex, monotonic preferences consumes non-negative amounts of x_1 and x_2.

   a.) If $u(x_1, x_2) = x_1^{\alpha} x_2^{\frac{1}{2}-\alpha}$ represents those preferences, what restrictions must there be on the value of parameter α ? Explain.

   b.) Given those restrictions, calculate the Marshallian demand functions.

   Solution

   a.) Monotonicity requires $\partial u/\partial x_1 = \alpha x_1^{\alpha-1} x_2^{\frac{1}{2}-\alpha} \geq 0 \Rightarrow \alpha \geq 0$ and $\partial u/\partial x_2 = \left(\frac{1}{2}-\alpha\right) x_1^{\alpha} x_2^{-\frac{1}{2}-\alpha} \geq 0 \Rightarrow \alpha \leq \frac{1}{2}$. Because the utility function is homogeneous of degree ½, it is strictly concave, hence also quasi-concave and quasi-concave functions have convex superior sets (i.e., preferences are convex). So no further restrictions on α are required.

   b.) $x_1 = \dfrac{2\alpha y}{p_1}$, $x_2 = \dfrac{(1-2\alpha) y}{p_2}$.

4. In a two-good case, show that if one good is inferior, the other must be normal.

   Solution

   $$p_1 x_1 + p_2 x_2 = y$$

   $$p_1 \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} = 1$$

   $$\frac{\partial x_1}{\partial y} \frac{y}{x_1} \frac{p_1 x_1}{y} + \frac{\partial x_2}{\partial y} \frac{y}{x_2} \frac{p_2 x_2}{y} = 1$$

   $$\eta_1 s_1 + \eta_2 s_2 = 1$$

   where

   $$\eta_i = \frac{\partial x_i}{\partial y} \frac{y}{x_i}, s_i = \frac{p_i x_i}{y}; \sum_{i=1}^{2} s_i = 1; i = 1,2$$

   Without loss of generality, assume x_1 is the inferior good. Then, we must have $\eta_1 < 0$, which means that $\eta_2 > 0$ because 1>0. Thus, x_2 must be a normal good.

5. How would you determine whether the function

   $$X\left(p_x, p_y, I\right) = \frac{2 p_x I}{{p_x}^2 + {p_y}^2}$$

   could be a demand function for commodity x of a utility maximizing consumer with preferences defined over the various combinations of x and y? Is it a demand function?

   Solution

One has to check all the properties of the Marshallian demand function, as well as the negative semi-definiteness of the Slutsky matrix. The function above is not a Marshallian demand function because the $s_{11}\left(p_x, p_y, \mathrm{I}\right)$ entry of the Slutsky matrix

$s_{11}\left(p_x, p_y, \mathrm{I}\right) = \dfrac{2I}{p_x^{\,2} + p_y^{\,2}} > 0$. This entry is non-positive for a well-behaved Marshallian demand function.

6. A firm produces output y from two inputs (x_1, x_2) using the production function $y = f(x_1, x_2)$. The output price is given by $p(y)$, the price of input one is w_1 per unit and the price of input two is w_2 per unit. That is, if the firm sells y units of output, the price it receives per unit is $p(y)$. Assume that f: $\mathbf{R}_+^2 \rightarrow \mathbf{R}_+^1$ is strictly concave and increasing and that p: $\mathbf{R}_+^1 \rightarrow \mathbf{R}_+^1$ is decreasing and convex. Both f and p are twice differentiable. Note that this firm is a price taker in the input market; its choices do not affect the input prices (w_1,w_2).

   a.) Write the firm's profit maximization problem and profit function. Let $\pi(w_1,w_2)$ be the profit function.
   b.) Is the partial derivative of $\pi(w_1,w_2)$ with respect to w_i equal to (−1) times the firm's input demand function for input i? Explain.
   c.) Is $\pi(w_1,w_2)$ a convex function of (w_1,w_2)? Explain.
   d.) Now suppose that $f\left(x_1, x_2\right) = x_1^{\beta} x_2^{1-\beta}$ and that $p\left(y\right) = y^{-\alpha}$, where $1 > \beta > 0$ and $1 > \alpha > 0$ Find the optimal input demands and output supply.

Solution

a.)

The CMP is:

$$C(w_1, w_2, y) = \min_{z_1, z_2 \geq 0} w_1 z_1 + w_2 z_2 \quad \text{s.t.} \quad f(z_1, z_2) \geq y; \tag{1}$$

and the PMP is:

$$\pi(w_1, w_2) = \max_{x_1, x_2 \geq 0} \; p(f(x_1, x_2)) f(x_1, x_2) - C(w_1, w_2, f(x_1, x_2)) \tag{2}$$

$$= \max_{y \geq 0} \; p(y) y - C(w_1, w_2, y). \tag{3}$$

b.)

Yes. Let $x(w_1, w_2, y)$ be the conditional factor demand (the solution to (1)), $z(w_1, w_2)$ be the factor demand function (the solution to (2)), and $y(w_1, w_2)$ be the supply function (the solution to (3)). They are all functions if the objective functions in the two forms of the PMP are all strictly concave (we assume this here). Applying envelope theorem to (3)[1] and then to (1) (i.e. Shepard's lemma), we get, for $i = 1, 2$:

$$\begin{aligned} \frac{\partial \pi(w_1, w_2)}{\partial w_i} &= -\frac{\partial C(w_1, w_2, y(w_1, w_2))}{\partial w_i} \\ &= -z_i(w_1, w_2, y(w_1, w_2)) \\ &= -x_i(w_1, w_2). \end{aligned} \tag{4}$$

c.)

Yes. By differentiating (4) with respect to w_j, we obtain:

$$\frac{\partial^2 \pi}{\partial w_i \partial w_j} = -\frac{\partial^2 C}{\partial w_i \partial w_j},$$

for $i = 1, 2$, $j = 1, 2$. Since we know from the CMP (1) that the cost function C is concave in (w_1, w_2), π is convex in (w_1, w_2).

<u>OR</u> you may show the convexity of π directly.

d.)

It is much more convenient to work with the CMP and the PMP (3) than to tackle the PMP (2) directly. Now the CMP becomes:

$$C(w_1, w_2, y) = \min_{z_1, z_2 \geq 0} w_1 z_1 + w_2 z_2 \quad \text{s.t.} \quad z_1^{\beta} z_2^{1-\beta} = y.$$

As the Cobb-Douglas production function is strictly concave, it follows that the FOC's are sufficient and necessary for finding the solution. We can also exclude the possibility of corner solution by requiring $y > 0$. The FOC's are:

$$\begin{aligned} -w_1 + \lambda \beta z_1^{\beta-1} z_2^{1-\beta} = 0 \quad &\Rightarrow \quad w_1 = \lambda \beta \left(\frac{z_2}{z_1} \right)^{1-\beta}, \\ -w_2 + \lambda (1-\beta) z_1^{\beta} z_2^{-\beta} = 0 \quad &\Rightarrow \quad w_2 = \lambda (1-\beta) \left(\frac{z_1}{z_2} \right)^{\beta}. \\ &\Rightarrow \frac{w_1}{w_2} = \frac{\beta}{1-\beta} \frac{z_2}{z_1}, \\ &\Rightarrow z_2 = \frac{w_1}{w_2} \frac{1-\beta}{\beta} z_1. \end{aligned}$$

Plugging into the constraint, we can solve for the conditional factor demand functions:

$$z_1(w_1, w_2, y) = y\left(\frac{w_2}{w_1}\frac{\beta}{1-\beta}\right)^{1-\beta}$$
$$\Rightarrow\ z_2(w_1, w_2, y) = y\left(\frac{w_1}{w_2}\frac{1-\beta}{\beta}\right)^{\beta}.$$

Plugging into the objective function, we get the cost function $C(w_1, w_2, y) = ay$ where

$$a := w_1^{\beta}\left(\frac{\beta}{1-\beta}w_2\right)^{1-\beta} + w_2^{1-\beta}\left(\frac{1-\beta}{\beta}w_1\right)^{\beta}.$$

We now turn to the PMP in (3). Now the PMP becomes:

$$\pi(w_1, w_2) = \max_{y\geq 0} y^{1-\alpha} - ay.$$

Obviously, the Inada condition is satisfied (the first derivative tends to $+\infty$ as $y \to 0$.), so $y > 0$ at the optimum. As the objective function is strictly concave ($\because 0 < \alpha < 1$), the FOC gives the solution:

$$(1-\alpha)y^{-\alpha} - a = 0$$
$$\Rightarrow\ y(w_1, w_2) = \left(\frac{1-\alpha}{a}\right)^{\frac{1}{\alpha}},$$

which is the supply function. The input demand functions are given by:

$$x_1(w_1, w_2) = z_1(w_1, w_2, y(w_1, w_2)) = \left(\frac{1-\alpha}{a}\right)^{\frac{1}{\alpha}}\left(\frac{w_2}{w_1}\frac{\beta}{1-\beta}\right)^{1-\beta},$$
$$x_2(w_1, w_2) = z_2(w_1, w_2, y(w_1, w_2)) = \left(\frac{1-\alpha}{a}\right)^{\frac{1}{\alpha}}\left(\frac{w_1}{w_2}\frac{1-\beta}{\beta}\right)^{\beta}.$$

7. Consider a competitive firm with a well-behaved production function $f(x)$ that converts an input x into a product q. The market price of the product is p and the price of the input is w. Derive the relationship between the curvature of the production function, i.e., f_{xx} and the elasticity of the product supply curve.

Solution

A competitive cost-minimizing food producer solves

$$\min_{\{x\}} C = wx,\ s.t.\ f(x) = q \tag{A1.1}$$

The properties of $f(\cdot)$, specified in the text, guarantee that it has an inverse, h, such that $f^{-1}(q) = h(q) = x$. The cost of production can thus be written as $C = wh(q)$. The firm equalizes the marginal cost to the output market price

$$MC = dC/dq = wh_q = p \tag{A1.2}$$

Totally differentiating equation (A1.2) and rearranging, we obtain

$$dq/dp = 1/(wh_{qq}) \tag{A1.3}$$

By Inverse Function Theorem, we have

$$h_q(q) = 1/f_x(h(q))$$

or more succinctly

$$h_q = 1/f_x \tag{A1.4}$$

Differentiating both sides of (A1.4) with respect to q and rearranging yields

$$h_{qq} = -\frac{1}{f_x^2}(f_{xx})(h_q) = -\frac{1}{f_x^2}(f_{xx})\frac{1}{f_x} = -\frac{f_{xx}}{f_x^3} \tag{A1.5}$$

The supply elasticity of a product is defined as

$$\eta_q^S = (dq/dp)(p/q) \tag{A1.6}$$

Combining the relationships (A1.2) to (A1.6), we obtain

$$f_{xx} = -f_x^2/\eta_q^S f \tag{A1.7}$$

8. Given the production function $f(x_{1,}x_{2,}) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2$, calculate the profit-maximizing demand and supply functions, and the profit function. For simplicity assume an interior solution. Assume that $\alpha_i > 0$.

   Solved in class

9. Corn (C) is produced from labor (L) using a decreasing returns to scale technology of the form $C = AL^{\varepsilon}$, where A is a scale parameter and $\varepsilon \in (0,1)$. How is the parameter ε related to the price elasticity of the corn supply curve?

   Solved in class