**Some Notes on Comparative Statics**
(drawn from Varian, and Perloff and others)

We will often need to use a simple model to determine the theoretical impacts of policy changes on behavior. You were given the tools to derive such comparative statics when you were taught the implicit function theorem and the envelope theorem. In general, we take the set of all equations that define behavior, and then totally differentiate with respect to both the exogenous factors and the decision variables. Here I provide several simple examples of how comparative statics may be used.

**Single Variable**

Suppose a monopolist solves

$$\max_q qD^{-1}(q) - C(q) - tq\ ,$$

where $q$ is the quantity produce, $D^{-1}(q)$ is price corresponding to a quantity of demanded $q$, $C(\cdot)$ is the production cost function and $t$ is a tax levied on production. The first order condition for this problem is

(1) $$D^{-1}(q) + qD^{-1}{}'(q) - C'(q) - t = 0\ ,$$

and the second order condition is given by

(2) $$2D^{-1}{}'(q) + qD^{-1}{}''(q) - C''(q) < 0\ .$$

In order to determine the impact of increasing the tax on quantity produced, we totally differentiate (1) and find

$$\left\{2D^{-1}{}'(q) + qD^{-1}{}''(q) - C''(q)\right\} dq - dt = 0\ ,$$

which can be solved as

(3) $$\frac{dq}{dt} = \frac{1}{\left\{2D^{-1}{'}(q) + qD^{-1}{''}(q) - C''(q)\right\}}.$$

We generally conduct comparative static analysis in order to determine the sign of behavioral change. In this case, the derivative in (3) must be negative. To see this note that the denominator is exactly the second order derivative of profit, and is required by (2) to be negative. This is the simplest version of a comparative static – uni-dimensional, with a clear sign imposed by the second order condition.

A similar example can be derived with the competitive firm. The competitive firm solves

$$\max_q qp - C(q) - tq,$$

with first order condition

$$p - C'(q) - t = 0,$$

and second order condition

$$-C''(q) < 0.$$

Totally differentiating the first order conditions with respect to the choice variable ($q$) and the policy variable ($t$) yields

$$-C''(q)\,dq - dt = 0,$$

or

$$\frac{dq}{dt} = -\frac{1}{C''(q)} < 0.$$

**Multi-variable**

Let a competitive market be represented by

(4) $$Q = D(p, y)$$

$$Q = S(p),$$

where $D$ is quantity demanded, $p$ is price, $y$ is income and $S$ is quantity supplied. As is normal, we assume $D_p < 0, D_y > 0, S_p > 0$. The equilibrium behavior is given by

(5) $$D(p, y) = S(p) = Q^*.$$

To determine the impact of changing income, we can totally differentiate (5) with respect to one behavioral variable ( $p$ ) and the exogenous variable ( $y$ ) to obtain

$$D_p dp + D_y dy = S_p dp,$$

or

$$\frac{dp}{dy} = \frac{D_y}{\left(S_p - D_p\right)}.$$

This must be positive. To see this note that the numerator is positive and denominator is positive. Further, totally differentiating (4) with respect to $Q, p, y$ obtains

$$dQ = D_p dp + D_y dy.$$

Thus,

$$\frac{dQ}{dy} = D_p \frac{dp}{dy} + D_y = \frac{D_p D_y}{\left(S_p - D_p\right)} + D_y = \frac{D_p D_y + \left(S_p - D_p\right) D_y}{\left(S_p - D_p\right)} = \frac{S_p D_y}{\left(S_p - D_p\right)}.$$

Again, this must be positive. A simpler way to approach a multivariate problem is to use matrix notation. In this case, we can represent the total differential of the supply and demand system with respect to all behavioral variables ( $Q, p$ ) and all exogenous variables ( $y$ ) as

$$\begin{bmatrix} 1 & -D_p \\ 1 & -S_p \end{bmatrix} \begin{bmatrix} dQ \\ dp \end{bmatrix} = \begin{bmatrix} D_y \\ 0 \end{bmatrix} dy.$$

We can then solve using Cramer's rule

$$\frac{dQ}{dy} = \frac{\begin{vmatrix} D_y & -D_p \\ 0 & -S_p \end{vmatrix}}{\begin{vmatrix} 1 & -D_p \\ 1 & -S_p \end{vmatrix}} = \frac{-S_p D_y}{-S_p + D_p} = \frac{S_p D_y}{S_p - D_p},$$

$$\frac{dp}{dy} = \frac{\begin{vmatrix} 1 & D_y \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -D_p \\ 1 & -S_p \end{vmatrix}} = \frac{-D_y}{-S_p + D_p} = \frac{D_y}{S_p - D_p}.$$

A more complicated example comes from the utility maximization model. Suppose an individual solves

$$\max U(q_1, q_2) = q_1 q_2,$$

subject to

$$p_1 q_1 + p_2 q_2 = y.$$

Then, we can write the LaGrangian as

$$L = q_1 q_2 + \lambda(y - p_1 q_1 + p_2 q_2).$$

The resulting first order conditions are

$$q_2 - \lambda p_1 = 0,$$

$$q_1 - \lambda p_2 = 0,$$

$$y - p_1 q_1 - p_2 q_2 = 0.$$

Totally differentiating with respect to all endogenous variables ( $q_1, q_2, \lambda$ ) and exogenous variables ( $p_1, p_2, y$ ) obtains

$$dq_2 - p_1 d\lambda = \lambda dp_1$$

$$dq_1 - p_2 d\lambda = \lambda dp_2,$$

$$-p_1 dq_1 - p_2 dq_2 = -dy + q_1 dp_1 + q_2 dp_2.$$

For convenience we tend to write the exogenous variables on the right hand side of equations and the endogenous variables on the left hand side. The exogenous changes will eventually be the denominator of our derivative. We can write this in matrix form as

$$\begin{bmatrix} 0 & 1 & -p_1 \\ 1 & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ q_1 & q_2 & -1 \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ dy \end{bmatrix}.$$

The determinant of the matrix on the far left is

$$\begin{vmatrix} 0 & 1 & -p_1 \\ 1 & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = 2p_1 p_2.$$

Hence, by Cramer's rule

$$\frac{dq_1}{dp_1} = \frac{\begin{vmatrix} \lambda & 1 & -p_1 \\ 0 & 0 & -p_2 \\ q_1 & -p_2 & 0 \end{vmatrix}}{2p_1 p_2} = \frac{p_2(-\lambda p_2 - q_1)}{2p_1 p_2} = \frac{-\lambda p_2 - q_1}{2p_1} < 0$$

$$\frac{dq_1}{dp_2} = \frac{\begin{vmatrix} 0 & 1 & -p_1 \\ \lambda & 0 & -p_2 \\ q_1 & -p_2 & 0 \end{vmatrix}}{2p_1 p_2} = \frac{-q_1 p_2 + \lambda p_1 p_2}{2p_1 p_2} = -\frac{q_1}{2p_1} + \frac{\lambda}{2} = 0 \text{ (from first ordercondition)}$$

$$\frac{dq_1}{dy} = \frac{\begin{vmatrix} 0 & 1 & -p_1 \\ 0 & 0 & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix}}{2p_1 p_2} = \frac{p_2}{2p_1 p_2} = \frac{1}{2p_1} > 0$$

$$\frac{dq_2}{dp_1} = \frac{\begin{vmatrix} 0 & \lambda & -p_1 \\ 1 & 0 & -p_2 \\ -p_1 & q_1 & 0 \end{vmatrix}}{2p_1p_2} = \frac{\lambda p_1 p_2 - p_1 q_1}{2p_1p_2} = \frac{\lambda}{2} - \frac{q_1}{2p_2} = 0 \text{ (from first order condition)}$$

$$\frac{dq_2}{dp_2} = \frac{\begin{vmatrix} 0 & 0 & -p_1 \\ 1 & \lambda & -p_2 \\ -p_1 & q_2 & 0 \end{vmatrix}}{2p_1p_2} = \frac{-p_1\left(q_2 + \lambda p_1\right)}{2p_1p_2} = \frac{-\lambda p_1 - q_2}{2p_2} < 0$$

$$\frac{dq_2}{dy} = \frac{\begin{vmatrix} 0 & 0 & -p_1 \\ 1 & 0 & -p_2 \\ -p_1 & -1 & 0 \end{vmatrix}}{2p_1p_2} = \frac{p_1}{2p_1p_2} = \frac{1}{2p_2} > 0$$

$$\frac{d\lambda}{dp_1} = \frac{\begin{vmatrix} 0 & 1 & \lambda \\ 1 & 0 & 0 \\ -p_1 & -p_2 & q_1 \end{vmatrix}}{2p_1p_2} = \frac{-q_1 - \lambda p_2}{2p_1p_2} < 0$$

$$\frac{d\lambda}{dp_2} = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & \lambda \\ -p_1 & -p_2 & q_2 \end{vmatrix}}{2p_1p_2} = \frac{-q_2 - \lambda p_1}{2p_1p_2} < 0$$

$$\frac{d\lambda}{dy} = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -p_1 & -p_2 & -1 \end{vmatrix}}{2p_1p_2} = \frac{1}{2p_1p_2} > 0$$

It may be useful to use programs such as Maple to solve these equations symbolically. Often you will not be able to determine a sign and must instead explore the factors that alter the sign of the comparative static. Finally, it is essential when writing an article that you provide some intuition for each of the presented comparative statics. Your audience will not believe these until you can give graphs and a story to accompany them.