Advanced Microeconomics

P. v. Mouche

Exercises 3

Exercise 1 *Given the following bimatrix game:*

$$\begin{pmatrix} 3;8 & 4;8 & 2;3 \\ 1;7 & 2;6 & 8;1 \\ 3;4 & 4;4 & 2;2 \\ 1;1 & 1;-1 & 1;-1 \end{pmatrix}.$$

a. *Determine the best reply correspondences.*

b. *Determine the dominant strategies and the strictly dominant strategies.*

c. *Determine the strategy profiles that survive the procedure of elimination of strongly dominated strategies.*

d. *Determine the Nash equilibria.*

e. *Determine the strongly Pareto-efficient strategy profiles and the weakly Pareto-efficient strategy profiles.*

Exercise 2 *Consider the bimatrix game*

$$\begin{pmatrix} 5;12 & 0;0 \\ 0;0 & 10;4 \end{pmatrix}.$$

a. *Determine, in case of pure strategies, the best reply correspondences, and if it/they exist, the strictly dominant Nash equilibrium, the iterative not strongly dominated equilibrium and the Nash equilibria.*

b. *Determine, in case of mixed strategies, the best reply correspondences and the Nash equilibria.*

Exercise 3 *Consider a game in strategic form with two players. The game is called 'strictly competitive' if for all strategy profiles (x_1, x_2) and (y_1, y_2)*

$$f_1(x_1, x_2) \geq f_1(y_1, y_2) \Leftrightarrow f_2(x_1, x_2) \leq f_2(y_1, y_2).$$

The game is called a 'constant-sum game' if there exists a number $c \in \mathbb{R}$ such that for each strategy profile (x_1, x_2) it holds that $f_1(x_1, x_2) + f_2(x_1, x_2) = c$.

a. *Show that each constant-sum game is strictly competitive.*

b. *Show that in a strictly competitive game each strategy profile is strongly Pareto efficient.*

Exercise 4 *Consider a homogeneous Cournot-oligopoly, i.e. a game in strategic form where each player i has as strategy set $X_i = [0, m_i]$ with $m_i > 0$ and a payoff function f_i of the form*

$$f_i(x_1, \ldots, x_n) = p(x_1 + \cdots + x_n)x_i - c_i(x_i),$$

where $p : [0, m_1 + \cdots + m_n] \to \mathbb{R}$ and $c_i : X_i \to \mathbb{R}$. Here x_i is called 'production level' of firm i. f_i is called 'profit function' of firm i, p 'inverse demand function' and c_i 'cost function' of firm i. A Nash equilibrium of this game also is called cournot equilibrium.

Suppose that p is decreasing, concave and twice differentiable and that each c_i is convex and twice differentiable.

a. Prove that each conditional profit function is concave.

b. Prove, using the Nikaido-Isoda theorem, that the game has at least one Cournot equilibrium.

Exercise 5 *Make Exercise 7.4 (on strong domination) from the text book.*

Short solutions.

Solution 1 a. $R_1(1) = \{1,3\}$, $R_1(2) = \{1,3\}$, $R_1(3) = \{2\}$, $R_2(1) = \{1,2\}$, $R_2(2) = \{1\}$, $R_2(3) = \{1,2\}$, $R_2(4) = \{1\}$.

b. Dominant strategies for player 1: do not exist.

Dominant strategies for player 2: the first.

Strictly dominant strategies: do not exist.

c. Step 1:

$$\begin{pmatrix} 3;8 & 4;8 \\ 1;7 & 2;6 \\ 3;4 & 4;4 \end{pmatrix}.$$

Step 2:

$$\begin{pmatrix} 3;8 & 4;8 \\ 3;4 & 4;4 \end{pmatrix}.$$

d. They are $(1,1)$ (i.e. row 1 and column 1), $(1,2), (3,1), (3,2)$.

e. Strongly: (1,2) (2,3). Weakly: (1,1), (1,2) (2,3), (3,2).

Solution 2 a. $R_1(1) = \{1\}$, $R_1(2) = \{2\}, R_2(1) = \{1\}$, $R_2(2) = \{2\}$.

No strictly dominant equilibrium and no iterative not strongly dominated equilibrium.

Nash equilibria: $(1,1)$ and $(2,2)$.

b. Expected payoff functions: $\overline{f}_1(p;q) = (15q-10)p + 10 - 10q$ and $\overline{f}_2(p;q) = (16p-4)q + 4 - 4p$.

Best reply correspondences: $\overline{R}_1(q) = \begin{cases} \{1\} \text{ if } q > 2/3, \\ [0,1] \text{ if } q = 2/3, \\ \{0\} \text{ if } q < 2/3 \end{cases}$ and $\overline{R}_2(p) = \begin{cases} \{1\} \text{ if } p > 1/4, \\ [0,1] \text{ if } p = 1/4, \\ \{0\} \text{ if } p < 1/4. \end{cases}$

Nash equilibria: $p = 1/4, q = 2/3$; $p = 0, q = 0$; $p = 1, q = 1$.

Solution 3 a. For a constant-sum game we have $f_1 + f_2 = c$. So $f_1(a,b) \geq f_1(c,d) \Leftrightarrow c - f_2(a,b) \geq c - f_2(c,d) \Leftrightarrow f_2(a,b) \leq f_2(c,d)$. Thus the game is strictly competitive.

b. By contradiction suppose $\mathbf{a}$ is strongly Pareto-inefficient. Then there exists a Pareto improvement $\mathbf{b}$ of $\mathbf{a}$. So then there exists a player, say 1, who at $\mathbf{b}$ has a greater payoff, and the other player does not have a smaller payoff. Thus $f_1(\mathbf{b}) > f_1(\mathbf{a})$ and $f_2(\mathbf{b}) \geq f_2(\mathbf{a})$. As the game is strictly competitive, we have $f_1(\mathbf{b}) \leq f_1(\mathbf{a})$, a contradiction.

Solution 4 a. Consider a player i. Fix a strategy of each other players. Let a be the sum of these strategies. For this situation the conditional payoff function is

$$g(x_i) = p(x_i + a)x_i - c_i(x_i).$$

Therefore

$$g''(x_i) = p''(x_i + a)x_i + 2p'(x_i + a) - c_i''(x_i).$$

As $p'' \leq 0, p' \leq 0$ and $c_i'' \leq 0$, it follows that $g'' \leq 0$. Thus g is concave.

b. Having part a, apply the Nikaido-Isoda equilibrium existence result (from the slides).

Solution 5 We consider the bimatrix game $(A;B)$ with $B = \begin{pmatrix} 0 & -3 & -4 \\ 4 & 5 & 8 \end{pmatrix}$. Then the second pure strategy of player 2 is not strongly dominated by a pure strategy. But for each mixed strategy (p_1, p_2) of player 1 we have $f_2((p_1,p_2),(1/2,0,1/2)) = 6 - 8p_1 > 5 - 8p_1 = f_2((p_1,p_2),(0,1,0))$. Therefore the second pure strategy of player 2 is strongly dominated by his mixed strategy $(1/2,0,1/2)$.