Advanced Microeconomics

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Exercises 2

Exercise 1 *Prove that a fully cooperative strategy profile is strongly Pareto efficient.*

Exercise 2 *Determine which of the following bimatrix games are a prisoner's dilemma.*

a. $\begin{pmatrix} 3;-1 & 3;1 & 6;0 \\ 1;0 & 3;1 & 6;0 \\ 2;2 & 4;1 & 8;2 \end{pmatrix}$.

b. $\begin{pmatrix} 1;0 & 3;1 & 6;0 \\ 2;1 & 4;1 & 8;1 \end{pmatrix}$.

c. $\begin{pmatrix} 6;1 & 3;1 & 1;5 \\ 2;4 & 4;2 & 2;3 \\ 5;1 & 6;1 & 5;2 \end{pmatrix}$.

d. $\begin{pmatrix} -1;-1 & 2;0 \\ 0;2 & 3;3 \end{pmatrix}$.

e. $\begin{pmatrix} 2;2 & -1;3 \\ 3;-1 & 0;0 \end{pmatrix}$.

Exercise 3 *Answer the following true/false questions concerning bimatrix games.*

a. A bimatrix game concerns a game with two players.

b. Each bimatrix game has at least one Nash equilibrium.

c. Each bimatrix game has a strictly dominant strategy.

d. Each bimatrix game has a fully cooperative strategy profile.

e. Each bimatrix game has a weakly Pareto efficient strategy profile.

f. Each fully cooperative strategy profile is weakly Pareto efficient.

e. A strictly dominant strategy is fully cooperative.

f. A prisoners' dilemma game has a Nash equilibrium.

g. It is impossible that a weakly Pareto inefficient strategy profile is a Nash equilibrium.

h. A Nash equilibrium is a strategy profile that consists of strategies of the players' that they like the most.

Exercise 4 *The following true/false questions deal with the bimatrix game*

$$\begin{pmatrix} 3;6 & 6;5 & 7;-3 \\ -6;2 & 5;3 & 5;4 \end{pmatrix}.$$

a. *The row-player has* 2 *strategies.*

b. *There are* 6 *strategy profiles.*

c. *The strategy profile* $(1,1)$ *is a Nash equilibrium.*

d. *The row-player has a strictly dominant strategy.*

e. *There is a weakly Pareto inefficient nash equilibrium.*

f. *The column-player has a strictly dominant strategy.*

g. *This game is a prisoners' dilemma.*

h. *Playing row* 1 *and column* 3 *is a fully cooperative strategy profile*

i. *This game is a zero-sum game.*

j. $(1,2)$ *is a weakly Pareto efficient strategy profile.*

Exercise 5 *Consider the Hotelling game in the case* $m = 2$ *(so there are three vertices) and* $w = 1$ *(i.e. inelastic case). Determine the Nash equilibria of this game*

a. *Represent this game as* 3×3*-bi-matrix game with at the first row strategy* 0 *for player* 1*, at the second row strategy* 1 *for player* 1*, etc.*

b. *Determine the Nash equilibria, the strongly Pareto efficient strategy profiles and the weakly Pareto efficient strategy profiles.*

Exercise 6 *Again consider the Hotelling Game with sites* $0, 1, \ldots, m$*. Suppose* m *is even.*

a. *Show that for the payoff function* f_1 *of player* 1

$$f_1(x_1, x_2) := \begin{cases} \frac{x_1+x_2+1}{2} \text{ if } x_1 < x_2, \\ \frac{m+1}{2} \text{ if } x_1 = x_2, \\ m+1-\frac{x_1+x_2+1}{2} \text{ if } x_1 > x_2 \end{cases}$$

b. *Show that* $(m/2, m/2)$ *is a Nash equilibrium.*

Short solutions.

Solution 1 We prove this by contradiction. So suppose $\mathbf{x}$ is fully cooperative and $\mathbf{x}$ would not be strongly Pareto efficient. Then there exists a pareto improvement $\mathbf{y}$ of $\mathbf{x}$. In $\mathbf{y}$ the sum of payoffs is greater than in $\mathbf{x}$. This is a contradiction with $\mathbf{x}$ being fully cooperative.

Solution 2 Only the game in e is a prisoner's dilemma game.

Solution 3 aT bF cF dT eT fT gF hT iF jF.

Some explanation. Concerning f (each fully cooperative strategy profile is weakly Pareto efficient): suppose the strategy profile $\mathbf{x}$ is fully cooperative, meaning that the total payoff is maximal. If it would not be weakly Pareto efficient, then there is a strategy profile which is better for both players and thus leads to a greater payoff than in $\mathbf{x}$. (In fact each fully cooperative strategy profile even is strongly Pareto efficient. In order to see this modify the above reasoning in an appropriate way.)

Concerning e: as each bimatrix game has a fully cooperative strategy profile, part f implies that each bimatrix game has a weakly Pareto efficient strategy profile.

Solution 4 aT bT cT dT eF fF gF hF iF jT.

Solution 5 a.

$$\begin{pmatrix} 3/2; 3/2 & 1; 2 & 3/2; 3/2 \\ 2; 1 & 3/2; 3/2 & 2; 1 \\ 3/2; 3/2 & 1; 2 & 3/2; 3/2 \end{pmatrix}.$$

b. There is a unique Nash equilibrium: the strategy profile $(2, 3)$, i.e. (vertex 1, vertex 1).

c. Each strategy profile is strongly Pareto efficient and weakly Pareto efficient (and even fully cooperative).

Solution 6 a. Make a figure and count the contributions. In doing so, ditinguish between $x_1 + x_2$ even and $x_1 + x_2$ odd.

b. We have to show that $f_1(x_1, m/2) \leq f_1(m/2, m/2)$ for all x_1 and that $f_2(m/2, x_2) \leq f_2(m/2, m/2)$ for all x_2. We prove here the first statement; the second follows in the same way.

For $x_1 = m/2$, the statement is clear. For $x_1 < m/2$, we have, using part a, $f_1(x_1, m/2) = \frac{x_1 + \frac{m}{2} + 1}{2} = \frac{x_1}{2} + \frac{m}{4} + \frac{1}{2} < \frac{m}{4} + \frac{m}{4} + \frac{1}{2} = \frac{m+1}{2} = f_1(\frac{m}{2}, \frac{m}{2})$. And for $x_1 > m/2$, we have, using part a, $f_1(x_1, m/2) = m + 1 - \frac{x_1 + \frac{m}{2} + 1}{2} = m + 1 - \frac{x_1}{2} - \frac{1}{2} - \frac{m}{4} = \frac{3}{4}m + \frac{1}{2} - \frac{x_1}{2} > \frac{3}{4}m + \frac{1}{2} - \frac{m}{4} = \frac{m+1}{2} = f_1(\frac{m}{2}, \frac{m}{2})$.