# The Well Tempered Meantone ___ Extended

Johan Broekaert ; broekaert.devriendt@telenet.be

**Abstract :**

Fifths and major thirds beating rate characteristics of famous historical temperaments are analysed.

It appears that beating rate characteristics might be the real temperament determining factor for auditory keyboard tuning, mainly because beating rates are of main importance to interpreting musicians regarding colour, harmony and possible musical affects, and to auditory tuners because of quality and ease of tuning. It is not always clear whether published ratios, cents or comma's are deduced from theoretic calculations or from concrete results on monochord measurements or settings.

A theoretical optimal auditory tuneable well (circulating) temperament is defined. An almost identical practical applicable well temperament is deduced from it, and no single historical temperament fits better with the defined theoretical optimum.

A number of characteristics that can be related to J. S. Bach, are discussed.

**Keywords**

Baroque ; well temperament ; meantone ; interval ; comma ; beating ; harmonic ; ratio ; cent ; Bach

## [1] Preamble

It is quite common to hear from people that they are more satisfied with an auditory tuned musical keyboard, than with the same keyboard that was tuned using a tuning instrument.

The auditory tuning of a keyboard is mainly based on the control of interval beating rates. Inherent to this control, auditory tuning depends on the timbre of the instrument sound, the harmonic structure thus of the sound, because it are the harmonics that generate the beatings. The timbre of notes thus, has an impact on the settings of intervals.

A sound with a perfect harmonic structure holds all harmonics frequencies at integer multiples of the frequency of the first harmonic. This is not so for sounds generated by physical vibrators ; their sounds usually are slightly inharmonic. The higher frequency components of a physically generated sound are therefore named "***partials***" instead of "***harmonics***".

The auditory tuner evaluates the comprehensive beating of an interval, this is broader than the beating specific to only two mutual harmonics or partials, and inharmonicity factors import.

Tuning based on tuning instruments on the other hand, involves setting note pitches to predetermined values, based on commonly calculated and published interval ratios. The timbre of notes are of little influence on the settings of their pitches. Limited tuning quality improvements are possible though, setting octaves based on separate measurement of partials, but important musical intervals, such as fifths or some major thirds, most often remain defined on calculated ratios rather than measured mutual frequencies of partials.

More tuning quality differences can be observed, due to slight differences between note pitch calculations based on interval ratios, and note pitch calculations based on interval beating rates.

The commonly published and dominating factors with discussions on musical temperaments are probably the investigations on purity deviations of musical intervals, measured in ratios, cents or commas.

And still, musical interval beatings and their beating rates are probably more affecting musical factors to auditory tuners of keyboard musical instruments and interpreting musicians, than the impurity measurements in ratios, cents or commas on the other hand, that are often nothing more but rather abstract concepts to many musicians, not of direct use or interest when playing music, nor for auditory tuning.

More attention might therefore have to be paid to interval beating characteristics : beatings are often directly observable and undesired. Approximate auditory beating rate evaluations do not require any tool nor calculation, but a metronome can be of help to improve precision.

This paper is an attempt to confirm and elucidate the importance and practical applicability of beating rate evaluations in the determination of musical temperaments.

## [2] The "Reason" at early times (Baroque and earlier)

At Baroque time, sound measuring tools had limited possibilities and precision only, and decimal systems or fractions were not yet currently applied. In general, many differing systems existed for all kinds of measures, often based on duodecimal fractions : money, length, weight, time, . . . Still today, this type of measures is used in some countries, among those not the least developed. Derived measures, such as volume, surface or speed for example, are even more complex.

The physics of sounds was not known in depth : it was not commonly known or clear yet, that musical sounds are periodic, and consist of a sum of sinusoidal waves.

There was no standard decimal notation of fractions. Some early decimal notation system is described by S. Stevin (1586), and the decimal units or the application of commas or points probably became introduced by G. Rheticus (1542), B. Pitiscus (1613) and J. Napier (1614). The calculation of roots, trigonometric values, logarithms, . . . was made by hand and very laborious.

Because of the above, the tuning of a keyboard was done auditory, based on perceived beating rates, not using any instrument except a diapason. The beating rates could be evaluated on intervals, or could be set in comparison with a pre-set note on a monochord.

## [3] The auditory music keyboard tuning

The elementary basic concepts of musical temperaments, seen from the point of view of the interpreting musician and the auditory music keyboard tuner are discussed in this paragraph.

There is of course much more that can be written on this subject, see for example : "Le Clavier Bien Obtempéré" (Calvet, 2020).

### [3.1] Musical Timbre

Music consists of ordained periodic sounds.

Those sounds can be generated by vibrating three-dimensional objects such as bells, or two-dimensional objects such as drums, or one-dimensional ones such as strings or air columns in a pipe. All those sounds can be composed by a sum of sinusoidal vibrations. The relations between those sinusoidal vibrations can be rather complex for the three– and two–dimensional vibrators. The sounds of one–dimensional vibrators are periodic and have a simple structure.

Musical harmony is mainly based on pleasing correspondence of simultaneous sounds of one-dimensional vibrators.

### [3.1.1] The vibrating string

Fig. 1 displays a momentary image of an infinitesimal short segment of a string "$ds$ , suspended on the origin, and on a point at position "$x$" on the $x$–axis.

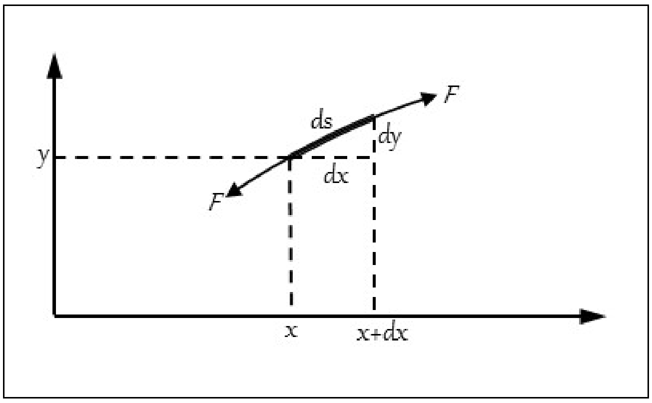


Fig. 1 : Segment of a vibrating string

The curve of the string under tension because of the forces "$F$" , leads to the generation of a vertical force component of the vectorial sum of those forces on the "$ds$" segment. Because of Newton's law, this force will also cause an acceleration of said segment.

With "$\mu$" standing for the specific mass of the string per unity of length, the equilibrium of the segment "$ds$" , consistent with analyses in physics textbooks, leads to :

$$\frac{F}{\mu} \cdot \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\partial^2 y(x,t)}{\partial t^2}$$

The obtained equation corresponds with a general unidimensional wave equation. The propagation speed of the wave is therefor :

$$v = \sqrt{F/\mu}$$

A simple sinusoidal solution of the above differential equation, complying with the requests of a node on positions "0" and "$L$" of the string, can be written as :

$$y_n(x,t) = A_n \cdot \sin\left(2\pi \frac{nv}{L} t + \varphi_n\right) \cdot \sin\left(2\pi \frac{nx}{L}\right) = A_n \cdot \sin(2\pi n f_1 t + \varphi_n) \cdot \sin\left(2\pi \frac{nx}{L}\right)$$

| | | |
|---|---|---|
| Whereby : | $A_n$ | is the amplitude of the wave |
| | $n$ | can be any positive integer number |
| | $\varphi_n$ | can be free chosen |
| | $f_1 = v/L$ | is the fundamental wave frequency |

The function $y_1(x,t)$ is called the fundamental, and the functions $y_n(x,t)$ for $n>1$ and integer, are called harmonics.

**[3.1.2] The vibrating closed pipe**

Fig. 2 displays a momentary image of an infinitesimal short gas segment "$dx$", at position "$x$" in a closed pipe under pressure "$P_0$", with segment "$S$", and a length "$L$".

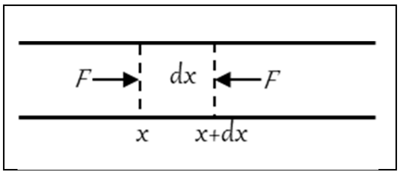

Fig. 2 : Segment of a vibrating pipe

An infinitesimal displacement of the segment "$dx$", generates an infinitesimal difference in pressure on the left and the right of that segment, due to a slight change of the left and the right gas volume, according the universal gas law "$PV^{\gamma}=nRT$", which in turn leads to an infinitesimal difference in the forces "$F$". Because of Newton's law, this force difference will also cause an acceleration of said segment.

With "$\mu$" standing for the specific mass of the gas per unity of volume, the equilibrium of the segment "$dx$", consistent with analyses in physics textbooks, leads to :

$$\frac{\gamma P_0}{\mu} \cdot \frac{\partial^2 P(x,t)}{\partial x^2} = \frac{\partial^2 P(x,t)}{\partial t^2}$$

The obtained equation corresponds with a general unidimensional wave equation. The propagation speed of the wave is therefor :

$$v = \sqrt{\gamma P_0 / \mu}$$

A simple sinusoidal solution of the above differential equation, complying with the requests of a pressure antinode on positions "0" and "$L$" of the pipe, can be written as :

$$yP_n(x,t) = A_n \cdot \sin\left(2\pi \frac{nv}{L} t + \varphi_n\right) \cdot \cos\left(2\pi \frac{nx}{L}\right) = A_n \cdot \sin(2\pi n f_1 t + \varphi_n) \cdot \cos\left(2\pi \frac{nx}{L}\right)$$

Whereby : $A_n$ is the amplitude of the wave
$n$ can be any positive integer number
$\varphi_n$ can be free chosen
$f_1 = v/L$ is the fundamental wave frequency

**[3.1.3] Pure Musical Intervals** (just and perfect musical intervals)

Based on the analysis of paragraphs 3.1.1 and 3.1.2, it can be said that instrumental music generated by strings or pipes, mainly consists of harmonic periodic sounds that consist of a sum of sine waves.

J. Fourier (1768–1830) developed mathematical evidence that ***any*** periodic function $F(t)$ consists of a sum of sine waves, whereby here again, ***the sine wave frequencies are INTEGER multiples of a basic frequency***.

$$F(t) = \sum_{n=1}^{\infty} [a_n \sin(2\pi n f t) + b_n \cos(2\pi n f t)] \quad \text{with} \quad n = \mathbb{N} \; ; \quad \text{and with :}$$

$$a_n = 2f \int_{-1/(2f)}^{1/(2f)} F\,(t) \sin(2\pi n f t)\, dt \quad \text{and} \quad b_n = 2f \int_{-1/(2f)}^{1/(2f)} F\,(t) \cos(2\pi n f t)\, dt$$

Purity of coincident musical sounds is usually desired. Coincident sounds are considered pure, if no beatings occur. A small musical interval impurity leads to a beating sound, because of ***the summation of mutual note partials or harmonics***, of almost equal frequency.
The sum of two sine waves is worked out in the formula below :

$$a \sin(2\pi f_a t) + b \sin(2\pi f_b t) = \sqrt{a^2 + b^2 + 2ab\cos[2\pi(f_a - f_b)t]} \times \cos\left(2\pi \frac{f_a + f_b}{2} t - \psi\right)$$

Hence, this sum corresponds to a single sine wave :

- of median frequency $(f_a + f_b)/2$
- with amplitude modulation from $(a - b)$ to $(a + b)$, at modulation frequency $(f_a - f_b)$
- with (low influence) phase modulation $\psi = \tan^{-1}\left[\frac{a+b}{a-b}\cot\left(2\pi\frac{f_a - f_b}{2}t\right)\right]$

Figure 3 displays the effect.

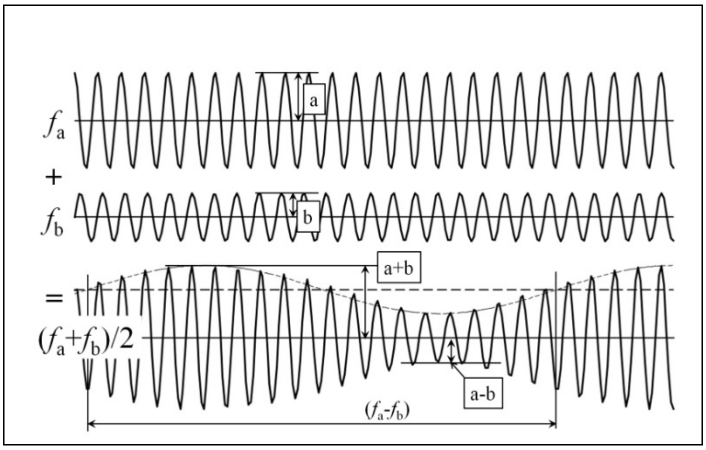


Fig. 3 Beating of two sine wave sounds

For example : the beating of an imperfect fifth − ratio ≈ 3/2 −, mainly results from the sum of the second harmonic of the upper note with the third harmonic of the lower note, – but also from any higher harmonic 2n of the upper note with any mutual higher harmonic 3n of the lower note –.
The lowest beating rate of a fifth can therefore be set as :

$$Beat_{\text{fifth}} = 2f_{\text{upper note}} - 3f_{\text{lower note}} = \mathrm{p}_1 f_2 - \mathrm{p}_2 f_1$$

This impurity measurement has been applied already by A. Kellner (1977), and is applicable for other intervals too, applying ***appropriate integer numbers for $p_1$ and $p_2$***.

All of the above demonstrates why important and pure musical intervals have ratios of (low) integer numbers.

Pure intervals lead to a pleasant and rich new sound, –a beautiful "consonance" (=harmony)–, because of reinforcement of common harmonics, and absence of beatings.

### [3.1.4] Observation

***Real physical vibrators may generate sounds that differ slightly from the above obtained theoretical models,*** due to physical characteristics deviating from the assumed “ideal” premises, leading to harmonics which frequencies deviate slightly from integer multiples of the fundamental frequency. For example : the stiffness of the string at its suspension points, inhomogeneity of strings, diameter of pipes . . . If so, the components for $n>1$ are called ***partials***. Inharmonicity can, for example, lead to stretched octaves for pianos (Railsback, 1938). Paintoux calculated a theoretic well tempered stretched piano tuning (Calvet 2020, Annexe 12, pp. 429 – 447).

**[3.2] Beating rate calculation of fifths and major thirds**

Auditory keyboard tuning is often initialized setting the notes on a scale from F3 to F4 (Calvet, 2020). The notes within this scale have rather low pitches and contain many harmonics, facilitating the tuning because of clear and low beating rates of intervals. Moreover, the major third on C will have the best ratio of all thirds, if all thirds have equal beating rate, because this major third has the highest pitch due to its high position on the F3 – F4 scale. The F3 – F4 scale also stands at a critical transition of string characteristics on a piano.
Beating rates within this scale can be calculated based on the equations tables 1 and 2.
The $q_{Note}$ and $p_{Note}$ symbols stand for the beating rates of fifths and major thirds.

| | | | |
|---|---|---|---|
| $q_F = 2C4 - 3F3$ | $q_C = 4G3 - 3C4$ | $q_G = 2D4 - 3G3$ | $q_D = 4A3 - 3D4$ |
| $q_A = 2E4 - 3A3$ | $q_E = 4B3 - 3E4$ | $q_B = 4F\#3 - 3B3$ | $q_{F\#} = 2C\#4 - 3F\#3$ |
| $q_{C\#} = 4G\#3 - 3C\#4$ | $q_{G\#} = 2Eb4 - 3G\#3$ | $q_{Eb} = 4Bb3 - 3Eb4$ | $q_{Bb} = 4F3 - 3Bb3$ |

Table 1 : calculation of fifths beating rate within the F3 – F4 scale

| | | | |
|---|---|---|---|
| $p_F = 4A3 - 5F3$ | $p_C = 4E4 - 5C4$ | $p_G = 4B3 - 5G3$ | $p_D = 8F\#3 - 5D4$ |
| $p_A = 4C\#4 - 5A3$ | $p_E = 8G\#3 - 5E4$ | $p_B = 4Eb4 - 5B3$ | $p_{F\#} = 4Bb3 - 5F\#3$ |
| $p_{C\#} = 8F3 - 5C\#4$ | $p_{G\#} = 2C4 - 5G\#3$ | $p_{Eb} = 8G3 - 5Eb4$ | $p_{Bb} = 4D4 - 5Bb3$ |

Table 2 : calculation of major thirds beating rate within the scale F3 – F4

**[3.3] Definition and comparison of temperaments**

**[3.3.1] Calculation of temperaments**

Quite some historical temperaments can be defined, based on the above formulas, if reliable, objective and historical data is available, regarding tuning instructions. Some very important historical temperaments are redefined in detail in paragraph 4 below, based on interval beating rate calculations. All calculations were worked out for a A4 = 440 diapason.

**[3.3.2] Comparison of temperaments**

Some discussed temperaments will have to be evaluated. Part of an evaluation can consist of the comparison of temperaments.

Although this paper relies on the interpretation of interval beating rates, rather than interval ratios, it is not adequate to use measurements in beatings per second (bps) as such for the comparison of temperaments : the beating rate is proportional to note pitch indeed, and therefor depends on the note pitch and the used diapason. Hence, a relative measure is preferable instead of an absolute measure in bps. An adequate and most common relative interval measure is the cent.

Characteristics of any temperament are unambiguously defined by either : the note pitches that depend on the set diapason, or relative measures such as semi–tones, fourths, fifths or sevenths (= 11 semi–tones). Fifths characteristics are most commonly applied ; just think of the circle of fifths.

Therefor, a comprehensive and objective comparison of temperaments can be based, for example, on a measure in cents of mutual fifths differences, by applying the formula below :

$$\Delta(\text{cent}) = \sqrt{\frac{\sum_{F,C,G,D,A,E,B,F\#,C\#,G\#,Eb,Bb}\left(fifth_{Note;temperament1} - fifth_{Note;temperament2}\right)^2}{12}}$$

## [4] Redefined temperaments

### [4.1] The equal temperament (12TET : twelve tone equal temperament)

A series of perfect fifths, ratio 3/2, leads to a Pythagorean temperament. It may be pentatonic (five consecutive fifths), diatonic ("through" consecutive fifths), or chromatic (twelve or more consecutive fifths). A chromatic Pythagorean temperament was described by Arnaut de (van) Zwolle (1400 – 1466), around 1440 to 1450, with all fifths perfect except a "wolf fifth" on B, also by Claas Douwes (1699) with a wolf fifth on G#. The wolf fifth impurity of the Douwes temperament holds – 16.86 . . . beatings per second (if A = 440). The Pythagorean temperament, was probably the dominating temperament applied in practice, until the begin of the 16–th century.

Alternative temperaments intervened, to avoid the disturbing beating of the wolf fifth. Among those the so called equal temperament. Vincenzo Galilei (c. 1520 – 1591) also, was probably among the firsts to apply an equal temperament.

Doubts can be expressed on which equal temperament was installed at that time. The 12TET is commonly accepted nowadays. But the required ratios were not yet known with high precision and were not simple to set precisely. The only available instrument at Baroque time to assist a tuner to set this temperament was the monochord.

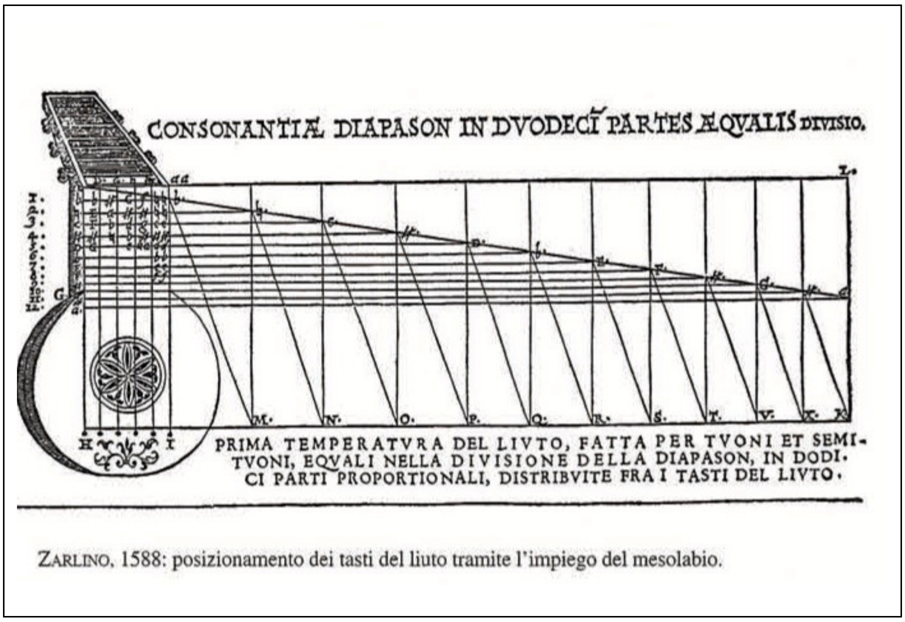


Fig. 4 : Historic drawing displaying how to construct an equal temperament

Early determination of equal temperament based on the twelfth root of "2" (= 1.05946 . . . ) was done by geometric construction ; see for example fig. 4. The triangles in the figure form a geometric size series, constructing equal geometric ratios. The angle of the hypotenuse, could not be defined mathematically, and had to be determined by iterated trial and error, in order to obtain the ratio

"2" after twelve steps. The obtained geometric ratios could be set as frets on a lute, or on a monochord to assist in auditory equal temperament tuning.

The exact mathematical 12TET equal temperament ratios became discussed by Zhu Zaiyu (1536 – 1611) and S. Stevin (1548 – 1620). The latter was probably the first European scientist to calculate the required ratios (ca. 1605). He calculated that string lengths on a monochord should be proportional to the figures displayed in table 3 (no decimals yet ! ). He applied some approximations in doing the calculations, and verification of the published figures shows some minor corrections for some of the obtained numbers are possible.

| C | C# | D | E*b* | E | F | F# | G | G# | A | B*b* | B | c |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10000 | 9438 | 8908 | 8409 | 7936 | 7491 | 7071 | 6674 | 6298 | 5944 | 5611 | 5296 | 5000 |
| OK | 9439 | 8909 | OK | 7937 | 7492 | OK | OK | 6300 | 5946 | 5612 | 5297 | OK |

Table 3 : required 12TET string length proportions on a monochord, according S. Stevin (+ minor corrections on second row)

Even today, it is not easy to set a 12TET temperament : a deviation of only 1 mm. on a string of 1 m. corresponds to a deviation of 1200 x $\log_2(1001/1000)$ = ~ 2 cents, and on top of that one has to count with the inharmonicity of any real physical string, tube or pipe. Precise electronic measurements show that measured string pitches might be unstable or vary in time. A fluctuation up to 0.22 Hz on the F3 note (173.87 to 174.09 Hz, this is ~ 2 cent), was observed by Calvet (2020, p. 282 – 284). He reports even up to 8 cents, due to a glitch in the measured values, but this might be due to minor deficiencies of the software measuring algorithms, probably FFT.

The German term for equal temperament is "Gleichstufige Temperatur" (equal temperament), but also "Gleichschwebende Temperatur" ( equally beating temperament). Therefore, one can think indeed of a temperament whereby all fifths have an equal beating rate instead of the equal and slightly reduced fifths ratio of the 12TET. A probable equal beating rate temperament was proposed by B. Fritz (Fritz 1756 ; Kroesbergen 2013, p. 16 – 20).
An equal beating temperament can easily be defined by setting ($q_{Note}$ : see table 1)

$$q_F = q_C = q_G = q_D = q_A = q_E = q_B = q_{F\#} = q_{C\#} = q_{G\#(Ab)} = q_{Eb} = q_{Bb}$$

Table 4 displays the obtained temperament, in comparison with the classical 12 TET

| | | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12TET | Pitches | 174.61 | 185.00 | 196.00 | 207.65 | 220.00 | 233.08 | 246.94 | 261.63 | 277.18 | 293.66 | 311.13 | 329.63 |
| | fifth beats | – 0.59 | – 0.63 | – 0.66 | – 0.70 | – 0.74 | – 0.79 | – 0.84 | – 0.89 | – 0.94 | – 0.99 | – 1.05 | – 1.12 |
| Fritz | Pitches | 174.66 | 185.05 | 196.00 | 207.69 | 220.00 | 233.15 | 247.00 | 261.60 | 277.18 | 293.60 | 311.13 | 329.60 |
| | fifth beats | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 | – 0.80 |

Table 4 : Comparison of the 12TET and Fritz equal temperaments (for A4 = 440) ; Δ(cent) = 0.48

It is clear that up to twelve differing equal beating rate temperaments can be obtained, depending on the chosen initial note for the twelve steps tuning scale, because a differing initial note goes at par with differing equations ; the equations of tables 1 and 2 are only valid for the chromatic F3 – F4 scale. ***The equal beating rate also alters with a factor two, for every octave step***, regardless the chosen initial note or diapason.

Auditory tuning instructions for an equal beating rate temperament are very simple : let all fifths within F3 to F4 beat at the prescribed equal beating rate of – 0.80 beatings/second (bps.). However, a very high tuning precision is required, in order to obtain an acceptable closure of the circle of fifths.

Corresponding auditory tuning instructions for the 12TET are more elaborate because of twelve differing beating rates, or, the tuning requires the use of a monochord or pitch measuring instruments, what goes at par with all already mentioned difficulties.

The above considerations, discussing auditory tuning based on equal beating rates, could put in doubt that early installed equal temperaments corresponded with the present "classical" 12TET. But it must be admitted : both equal temperaments are very comparable, with Δ(cent) = 0.48 cent only. The auditory differences are indistinguishable. The 12 TET mean beating rate (within the F3 − F4 scale), is − 0.83 . . . bps. ; this is only slightly more and quasi identical to the − 0.80 . . . bps. of an equal beating rate temperament.

**[4.2] The Quarter Comma Meantone Temperament**

Just major thirds, ratio 5/4 (= 1.25), are in general appreciated and desired by musicians and their audience. The quality of major thirds can determine the differing colours, characters or affects of differing temperaments. Hence, much importance is paid to the just major third. Just major thirds differ from the rather sharp ones of the Pythagorean tuning, having a ratio 81/64 (= 1.265625). The difference is the syntonic comma, with ratio 81/80. The 12TET major thirds also, are quite sharp, with a ratio $2^{4/12}$ (= 1.2599 . . .). Early considerations on the importance of the just major thirds, can, among others, be attributed to Ptolemaeus (c. 100 − c. 170).

The just major thirds became widely introduced and accepted because of the meantone temperament. This temperament was described in 1523 by P. Aaron and by Salinas (1577) according Zarlino. This meantone is based on a just major third on C that is divided in four equal halftones. This temperament is therefore called the "quarter (syntonic) comma meantone". All other notes emanate from just major thirds on the notes Eb, Bb, F, (C), G, D, A, E.
***Please notice : ALL allowed meantone keys* (B*b*, F, C, G, D, A,) *offer EXCELLENT and IDENTICAL HARMONY, the only difference between the allowed keys consists of a tonal height difference only.***
Therefore, modulation is musically completely free for meantone tuned instruments, as long as it remains within the allowed keys.
The meantone was probably the "dominating" temperament for Baroque music.

For the meantone also, one may wonder whether the C − E major third should be divided based on ratios, cents or commas, as is commonly assumed, presumably by means of a monochord or measuring instrument, or whether it should somehow have been based on beating rates ?

It is clear that the equal division in four half tones of the C − E major third, corresponds with four equal ratio impurity fifths (E − A, D − A, G − D, C − G) on which the major third is built. Therefore, a hypothetical alternative division might consist of building the just major third C − E, based on four fifths with equal beating rate (E − A, D − A, G − D, C − G), followed by the further installation of seven additional and desired just major thirds. This determination of notes also has the advantage it contains a direct link with the A note when initiating the tuning, but also the C note ; the latter imports if the C note is chosen as diapason, as was or is often done.
The initial note pitches can thus be obtained by solving following equations (see table 1 and 2) :

A pure major third on C : $p_C = 0$
and equal beating fifths on C, G, D, A : $q_C = q_G = q_D = q_A = q_{Note}$

The remaining notes are defined, by setting pure major thirds on the notes E*b*, B*b*, F, G, D, A, E :

$$p_{Eb} = p_{Bb} = p_F = p_G = p_D = p_A = p_E = 0$$

Table 5 displays the obtained pitches, pitch differences Δ–pitch (***in cents***), beating rates, and the comprehensive fifths difference Δ(cent) with the classical version :

| Meantone | | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| "classic" | Pitches | 176.00 | 183.90 | 196.77 | 205.61 | 220.00 | 235.40 | 245.97 | 263.18 | 275.00 | 294.25 | 314.84 | 328.98 |
| | fifth beats | − 1.64 | − 1.71 | − 1.83 | 12.85 | − 2.05 | − 2.19 | − 2.29 | − 2.45 | − 2.56 | − 2.74 | − 2.93 | − 3.06 |
| Equal beating | Pitches | 176.00 | 183.79 | 196.78 | 205.56 | 220.00 | 235.26 | 245.98 | 263.12 | 275.00 | 294.07 | 314.85 | 328.89 |
| | Δ–pitch | 0.00 | − 1.03 | 0.09 | − 0.43 | 0.00 | − 1.03 | 0.09 | − 0.43 | 0.00 | − 1.03 | 0.09 | − 0.43 |
| | fifth beats | − 1.77 | − 1.38 | − 2.21 | 13.03 | − 2.21 | − 1.77 | − 2.76 | − 2.21 | − 2.76 | − 2.21 | − 3.54 | − 2.76 |

Table 5 : Comparison of the "classic" meantone with the auditory tuned one (for A4 = 440) ; Δ(cent) = 0.43 ; Δ–pitch(***cent***)

This proposed alternative auditory tuning could be installed at much ease by any Baroque tuner, not using any measuring tool at all.

Precise tuning of the "classic" division of the just C − E third was only possible by means of a monochord, but this goes at par with quite intensive labour : for a monochord of length of 1000 mm., tuned on the note C, the movable bridge must be set at C# = 935.73 mm, D = 893.33 mm, Eb = 835.90 mm, and E = 800.00 mm. A deviation of only 1 mm corresponds to a pitch deviation of 2.16 . . . cents already. This is more than twice the maximum differences displayed at table 5 (1.03 cent). Doubts can therefore be expressed on the effective installation during the Baroque period, of the nowadays commonly published pitches of the meantone temperament, based on ratio calculations.

There is in fact no single practical reason not to tune the auditory way. The differences between both versions are practically and auditory not distinguishable.

### [4.3] **Well Temperaments** (circulating temperaments)

Well temperaments strive for best interval purities for the diatonic C–major, and acceptable purity for the other tonalities. A musical definition of well temperaments, based on the Werckmeister criteria, was elaborated by H. Kelletat (1960 ; 1981, p. 9) :

> << Well temperament means a mathematical–acoustic and musical–practical organization of the tone system within the twelve steps of an octave, so that impeccable performance in all tonalities is enabled, based on the extended just intonation (natural–harmonic tone system), while striving to keep the diatonic intervals as pure as possible.
> This temperament acts, while tied to given pitch ratios, as a thriftily tempered smoothing and extension of the meantone, as unequally beating half tones and as equal (equally beating) temperament. >> (see footnote for original German text [1])

There has been a general and very active quest on well temperaments during the Baroque period. Reason for this quest was that the meantone does not allow for acceptable musical modulation in all keys, because of the "wolf fifth" (usually on G#) and the associated four "harsh" major thirds.

Werckmeister was, among others, at the origin of the well temperament concept, and his well tempered Werckmeister III temperament (1691) became famous. A well temperament might have been requested for the reconstruction of the organ of the Lucca Cathedral (Italy), in 1473 (Devie 1990, p. 55).

Werckmeister is probably also the first to have used the "wohltemperiert" (well tempered) term in writing :

- (Werckmeister, 1681 ; Norback, 2002, p 18, fig. 2) : Title page : "…ein Clavier *wohl zu temperiren* und zo stimmen sei / …"
- (Werckmeister, 1686) :
  - Chapter 30, page 118 (erratically marked as 108) : "…/ wenn unser Clavier *wol temperiret* ist / …"
  - Page 120, 16–th rule : "…Wenn wir hingegen ein *wohl temperirtes* Clavier haben / …"
- (Werckmeister, 1691) :
  - Title page : "…/ und dergleichen *wol temperirt*…"
  - Chapter 22, page 61, 7–th rule : "…/ wenn nun ein Clavier *wohl temperiret* ist / …"
- (Werckmeister, 1698, p. 7) : "… / und wäre nicht *wohl temperiret* oder …"

Nowadays again, well temperaments (= circulating temperaments) have become a hot musical topic.

It is quite probable that the publications of H. Kelletat (1956, 1960, 1981, 1982) are at the origin of the present interest. A. Calvet (2020) also, in a more recent publication, such as H. Kelletat, offers considerations in width and in depth on the musical temperament and tuning topics, supported by historical aspects and profound explanation on how and why musical temperaments, intervals and interval beating rates have specific important characteristics. Calvet has treated in particular and depth also the aspect of musical interval beating, and the beating rates of many temperaments are well documented. He is also discussing the importance of required interval readjustments during the practice of piano tuning, because of inharmonicity of strings, or corrections for better distribution of beatings, and these corrections might therefore be due to desired auditory beating rate improvements leading to deviations from published pitches. Jobin (2005) also, mentions the necessity of interval readjustments.

**[4.3.1] Werckmeister** (1635−1706)

Werckmeister has published his tuning instructions, based on commas, see table 6 concerning Werckmeister III (1698, chap. 30, p. 78), his most applied and famous temperament. This temperament also, can be recalculated, based on beating rates, setting :

$q_C = q_G = q_D = q_B = q_{Note}$ and all others : $q_B = q_B = q_B = q_B = q_B = q_B = q_B = q_B = 0$

The obtained pitches and beatings are displayed in table 6.

| | | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Werckm. III | Pitches | 175.60 | 185.00 | 196.89 | 208.12 | 220.00 | 234.14 | 247.50 | 263.40 | 277.50 | 294.33 | 312.18 | 330.00 |
| | commas | 1/4 | 0 | 1/4 | 0 | 0 | 0 | 1/4 | 1/4 | 0 | 0 | 0 | 0 |
| | beatings | – 0.00 | 0.00 | – 2.00 | 0.00 | 0.00 | 0.00 | – 2.51 | – 2.67 | 0.00 | – 2.99 | 0.00 | 0.00 |
| beat rate Werckm. | Pitches | 175.61 | 185.00 | 196.94 | 208.13 | 220.00 | 234.14 | 247.50 | 263.41 | 277.50 | 294.16 | 312.19 | 330.00 |
| | beatings | 0.00 | 0.00 | – 2.49 | 0.00 | 0.00 | 0.00 | – 2.49 | – 2.49 | 0.00 | – 2.49 | 0.00 | 0.00 |

Table 6 : comparison between the "classic" Werckmeister III, and the "equal beating" Werckmeister III ; Δ(cent) = 0.52

The differences between the published and recalculated versions are minimal, and can very probably not be distinguished auditory.

**[4.3.2] Vallotti** (1697−1780)

The Vallotti temperament is characterized by equality of diminished diatonic fifths, all other fifths being perfect. It is part of the countless amount of well temperaments created at Baroque time. The diminished fifths lead to some rather improved but still not yet just diatonic major thirds.

For Vallotti also, a comparison is possible between the temperament based on equality of fifths in cents, and the one based on equality in beating rates. The "beating rate Vallotti" is easy to calculate, setting ($q_{Note}$ : see table 1) :

$q_F = q_C = q_G = q_D = q_A = q_E = q_{Note}$ and all others : $q_B = q_{F\#} = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb} = 0$

The obtained pitches are displayed in table 7.
For Vallotti too, the tuning procedure based on equal fifths beating rates is much simpler than the one based on ratios. The applicable fifths beating rate is −1.59 . . . bps. (within F3−F4)

| | | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Vallotti | Pitches | 175.40 | 184.79 | 196.44 | 207.89 | 220.00 | 233.87 | 246.38 | 262.51 | 277.18 | 294.00 | 311.83 | 329.26 |
| | beatings | – 1.19 | 0.00 | – 1.33 | 0.00 | – 1.49 | 0.00 | 0.00 | – 1.78 | 0.00 | – 1.99 | 0.00 | – 2.23 |
| Beat rate Vallotti | Pitches | 175.49 | 184.88 | 196.44 | 207.99 | 220.00 | 233.99 | 246.51 | 262.45 | 277.32 | 293.86 | 311.99 | 329.21 |
| | beatings | – 1.59 | 0.00 | – 1.59 | 0.00 | – 1.59 | 0.00 | 0.00 | – 1.59 | 0.00 | – 1.59 | 0.00 | – 1.59 |

Table 7 : comparison between the "classic" Vallotti, and an "equal beating" Vallotti ; Δ(cent) = 0.61

Both Vallotti scales are almost identical and are probably auditory indistinguishable.

**[4.3.3] Kirnberger** (1721–1783)

After two rejected proposals, Kirnberger proposed the Kirnberger III temperament, a commonly known well temperament, which is probably among the most famous ones. The Kirnberger III is characterized by a just major C − E third, and all fifths perfect, except those involved in building the C − E third and the one on F#.

Two versions exist : Kirnberger III, and Kirnberger III ungleich (Kelletat, 1981, p. 158, table 12). The "common" Kirnberger III version has ***equal impurity, expressed in ratios or cents***, for the fifths on C, G, D, A building the just C – E major third. The Kirnberger III ungleich version has unequal ratios for those fifths, but, surprisingly, within the C4 – C5 octave those have almost equal beating rates for G4, D4, A4 and C5 (not C4).

The Kirnberger III temperaments are among those that are easiest to set in case of auditory well tempered tuning : as soon as the just major third C – E is set, it is sufficient to set seven more perfect fifths, the longest chain of fifths containing only five of those (those on F, B*b*, E*b*, G#, and C#).
Redefinition of Kirnberger III is possible, setting (see tables 1 and 2) :

A pure major third on C : $p_C = 0$ and equal beating fifths on C, G, D, A : $q_C = q_G = q_D = q_A$

and perfect fifths on E, B, C#, G#, E*b*, B*b*, F : $q_E = q_B = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb} = q_F = 0$

Results are on display at table 8.
It should be noticed that the initial notes C4, G3, D4, A3, E4 for the "beating rate" Kirnberger III are identical to those of the "beating rate" meantone (see table 5).
The differences between the versions are probably auditory indistinguishable.

| | | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Kirnberger III | Pitches | 175.45 | 185.05 | 196.77 | 207.95 | 220.00 | 233.94 | 246.73 | 263.18 | 277.26 | 294.25 | 311.92 | 328.98 |
| | beatings | 0.00 | – 0.63 | – 1.83 | 0.00 | – 2.05 | 0.00 | 0.00 | – 2.45 | 0.00 | – 2.74 | 0.00 | 0.00 |
| Kirnb. III ungl | Pitches | 175.40 | 184.99 | 196.90 | 207.88 | 220.00 | 233.87 | 246.66 | 263.10 | 277.18 | 294.50 | 311.82 | 328.88 |
| | beatings | 0.00 | – 0.63 | – 1.70 | 0.00 | – 2.25 | 0.00 | 0.00 | – 1.70 | 0.00 | – 3.50 | 0.00 | 0.00 |
| Beating rate Kirnberger III | Pitches | 175.41 | 185.00 | 196.78 | 207.89 | 220.00 | 233.88 | 246.67 | 263.12 | 277.19 | 294.07 | 311.84 | 328.89 |
| | beatings | 0.00 | – 0.63 | – 2.21 | 0.00 | – 2.21 | 0.00 | 0.00 | – 2.21 | 0.00 | – 2.21 | 0.00 | 0.00 |

Table 8 : comparison between the "classic" Kirnberger III, Kirnberger III ungleich, and the recalculated one
Δ(cent) with Kirnberger III = 0.48 ; Δ(cent) with Kirnberger III ungleich = 1.17

### [4.3.4] Other Historic Temperaments

More historic temperaments can be redefined and analysed, if reliable, adequate and documented historic information is available, the early Baroque ones being the most interesting. Most consulted data originate from De Bie (2001, most temperaments are based on Barbour, except for recent ones).

See Appendix A for details on a number of recalculations.

**Observation :**

All obtained Δ(cent) display close fits between cent calculated and beating calculated temperaments, with mean fifths differences [this is = Δ(cent)] mostly around 0.5 cent, whereby the next best fitting historical temperaments, mostly differ more than 1 cent.

The above differences are low, in comparison with the differences between "classic" well temperaments and the 12 TET, usually around 2 cents. Differences can be as high as 11 cents (12TET / meantone) or even up to 24 cents (Pythagorean temperament with wolf fifth on G# / Salinas).

## [5] Well Temperament Optimisation

One might wonder for an optimal Well Temperament, holding an optimal C–major diatonic purity.

An objective criterion is required for the mathematical elaboration of such an optimum. But even so, it is not probable that such a criterion could be imposed musically, and musicians will probably not agree upon one : it is indeed sufficient to overlook the vast collection of well temperaments to get convinced about the musical subjectivity of the topic. Nevertheless, the availability of some optimum can be useful and so we make an attempt to define one.

### [5.1] Optimal C–major Diatonic Purity, for a lowest impurity, for fifths on C4, G3, D4, A3, E4 and major thirds on F3, C4, G3.

To develop an "optimal impurity" mathematical model temperament, an impurity measure is proposed, consisting of the quadratic sum of the impurities of the C–major fifths and major thirds : those are the main intervals controlled for tuning. Two "optimal impurity" temperaments are defined : one for cent calculations and one for beating rate calculations. See Appendix B.

Observation

Both obtained "optimal impurity" temperaments are very similar, as expected : Δ(cent) = 0.38.

Both temperaments have C–major diatonic fifths beating rates within their own scale, that are quite dissimilar : from – 0.77 to – 3.22 bps. This is not comfortable for auditory tuning.

**[5.2] Alternative scale, with BEST EQUALITY of BEATING RATE impurities, for fifths on C4, G3, D4, A3, E4 and major thirds on F3, C4, G3.**

In auditory tuning practice it is not possible to set a minimum beating rate, without application of high precision measuring tools. Those tools are required for high precision measurement of the beating rates of six fifths and three major thirds, their sum, and their minimum.

It is probably musically better, and also easier for auditory tuning, to estimate the equality of slow beating rates, rather than their minimum sum ; no tool at all is required to make a good estimation of beating rate equalities. But a mathematically exact beating rate equality is probably not possible : we have nine equations (beating rates of six fifths and three major thirds) for six variables (F, C, G, D, E, B). We can therefor only strive for a best possible equality.

Good equality means a small deviation from an average value "$M$". Beatings are normally negative on fifths (too small), and positive on thirds (too large). Normally the absolute average "$M$" therefore is :

$$M = \frac{-q_F - q_C - q_G - q_D - q_A - q_E + p_F + p_C + p_G}{9}$$

Taking the signs into account, the deviations from the mean beating are :

Fifths : $\Delta_{Fi;Note} = -q_{Note} - M$ Major Thirds : $\Delta_{T;Note} = p_{Note} - M$

The sum of the squares of these deviations becomes :

$$\sum \Delta^2_{Fi\ and\ T;Note} = \Delta^2_{Fi;F} + \Delta^2_{Fi;C} + \Delta^2_{Fi;G} + \Delta^2_{Fi;D} + \Delta^2_{Fi;A} + \Delta^2_{Fi;E} \quad + \quad \Delta^2_{T;F} + \Delta^2_{T;C} + \Delta^2_{T;G}$$

the elaboration of this sum in function of the notes leads to :

$$\begin{aligned} 81 \times \sum \Delta^2_{Fi\ and\ T;Note} = {} & 2718F_3^2 + 2934C_4^2 + 3726G_3^2 + 1044D_4^2 + 3240A_3^2 + 2124E_4^2 + 2592B_3^2 \\ & -1116F_3C_4 - 216F_3G_3 + 36F_3D_4 - 3132F_3A_3 + 180F_3E_4 \\ & -2376C_4G_3 + 72C_4D_4 + 216C_4A_3 - 2880C_4E_4 \\ & -864G_3D_4 + 324G_3A_3 + 540G_3E_4 - 3240G_3B_3 \\ & -1998D_4A_3 - 90D_4E_4 \quad - 1242A_3E_4 \quad - 1944E_4B_3 \end{aligned}$$

According advise from Prof. E. Amiot, the minimum of this expression, can be elaborated by calculation of the set of partial derivatives to the notes set to zero.
Table 9 displays the coefficients of the simplified partial derivatives set to zero :

| | F3 | C4 | G3 | D4 | E4 | B3 | = | A3 |
|---|---|---|---|---|---|---|---|---|
| ∂/∂F3 : | 151 | – 31 | – 6 | 1 | 5 | 0 | = | 87 |
| ∂/∂C4 : | – 31 | 163 | – 66 | 2 | – 80 | 0 | = | – 6 |
| ∂/∂G3 : | – 2 | – 22 | 69 | – 8 | 5 | – 30 | = | – 3 |
| ∂/∂D4 : | 2 | 4 | – 48 | 116 | – 5 | 0 | = | 111 |
| ∂/∂E4 : | 10 | – 160 | 30 | – 5 | 236 | – 108 | = | 69 |
| ∂/∂B3 : | 0 | 0 | – 5 | 0 | – 3 | 8 | = | 0 |

Table 9 : Calculation of diatonic notes for C – major

The obtained B and F pitches (see further table 10) impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{246.22}{175.67} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500545\ldots$$

The above ratio is slightly above perfection, but so little that it still can be acceptable within a well temperament. Further creation of an “optimal” well temperament therefor, only allows for even distribution of this minute obligate fifths enlargement over the six remaining fifths.

$$q_{Note} = q_B = q_{F\#} = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb}$$

| | F3 | F#3 | G3 | G#3 | A3 | B*b*3 | B3 | C4 | C#4 | D4 | E*b*4 | E4 | F4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_{Note}$ | 175.67 | 184.73 | 196.60 | 207.98 | 220.00 | 234.14 | 246.22 | 262.75 | 277.22 | 293.96 | 312.10 | 328.93 | 351.34 |
| $q_{Note}$ | – 1.52 | 0.26 | – 1.89 | 0.26 | – 2.15 | 0.26 | 0.26 | – 1.83 | 0.26 | – 1.87 | 0.26 | – 1.89 | – 3.04 |
| $p_{Note}$ | 1.65 | 12.91 | 1.89 | 11.07 | 8.90 | 5.13 | 17.30 | 1.97 | 19.23 | 8.06 | 12.30 | 19.23 | 3.30 |

Table 10 : scale with optimal beating rate equality of the of diatonic fifths and major thirds within C major

The average beating rate for C–major diatonic fifths and major thirds between F3 and F4, is – 1.85 . . . bps., with minor deviations only from this value (≤ 0,33).

Observation :
An ***equal cent-impurity*** model temperament can also be developed ; see Appendix C.

### [5.3] A possible “Optimal” Equal Beating Rate, in Practice

Both, the Pythagorean system with 11 perfect fifths as well as the meantone system with 8 just major thirds, hold an undesired wolf fifth. Such as the equal temperament was derived from the Pythagorean system, one can strive to derive a well temperament from the meantone (cfr. the well temperament definition, section 4.3). Tempering of the meantone wolf fifth has lead to a number of meantone variants, among those some holding less pure major thirds, such as Rameau (4 MT), Marpurg, (4 MT), Legros (3 and 2 MT), d’Alembert (1 MT), de Béthisy (1 MT) . . . or others with differing division of the syntonic comma, such as Salinas (1/3 comma), Zarlino (2/7 comma), Sauveur (1/5 comma), Romieu (1/6 and 1/7 comma), . . .

#### [5.3.1] A circulating tempered meantone holding less just major thirds (auditory tuneable)

##### [5.3.1.1] The natural diatonic notes of C–Major

Inspired by E. Jobin (2005), the tuning can be initiated, installing three just major thirds within the diatonic C–major ; one on C, and those on F and G to follow ; Jobin defined two just major thirds. All natural diatonic notes have thus become defined. Their pitches are equal to those that are already on display in table 5 (the equal beating meantone version).

##### [5.3.1.2] The altered notes of C–Major

The obtained B and F pitches impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{245.98}{176.00} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.501258\ldots$$

The above ratio is slightly above perfection, but so little that it can be acceptable still, within a well temperament. Further creation of a well temperament therefor, does no more allow for additional major thirds comparable to those already defined, while this leads to a further undesired and noticeable enlargement of the remaining fifths.

For ease of auditory tuning, but in line with classic Pythagorean tuning, the notes F#, C# and G# can become defined instead, by perfect fifths on B, F# and C# ; those perfect fifths have an extremely minor difference only, with the above determined 1.501258 . . . mean ratio :

$$q_B = q_{F\#} = q_{C\#} = 0$$

The remaining and obligate slightly increased fifths impurities can finally be equally distributed over the fifths on Ab(G#), E*b*, B*b*, in order to define the pitches of B*b* and E*b* :

$$q_{Ab} = q_{Eb} = q_{Bb}$$

Observation: the ease of tuning could also have been achieved instead, by setting perfect fifths on the flat notes A*b*, E*b*, B*b*, and slightly augmented fifths on the notes B, F# and C#. The above proposed arrangement has the advantage that non of the natural notes of the diatonic C–major holds an augmented fifth.

**[5.3.1.3] Obtained circular tempered meantone**

The collection of solutions leads to the scale, table 11 :

| | F3 | F#3 | G3 | G#3 | A3 | B*b*3 | B3 | C4 | C#4 | D4 | E*b*4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_{Note}$ | 176.00 | 184.48 | 196.78 | 207.55 | 220.00 | 234.26 | 245.98 | 263.12 | 276.73 | 294.07 | 311.93 | 328.89 |
| $q_{Note}$ | – 1.77 | 0.00 | – 2.21 | 1.23 | – 2.21 | 1.23 | 0.00 | – 2.21 | 0.00 | – 2.21 | 1.23 | – 2.76 |
| $p_{Note}$ | 0.00 | 14.60 | 0.00 | 14.73 | 6.91 | 5.00 | 17.83 | 0.00 | 24.36 | 5.53 | 14.61 | 15.89 |

Table 11 : Circular Tempered Meantone ; Δ(cent) = 1.39

As was observed already, the circulating tempered meantone holds natural notes identical to those of the beating rate defined meantone, table 5.

**[5.3.2] A WELL TEMPERED MEANTONE**
**with a different division of the syntonic comma (auditory tuneable)**

This paper, especially this paragraph, came about thanks to intense correspondence and close cooperation with A. Calvet, professional auditory tuner.

**[5.3.2.1] Tuning initiation**

Tuning history shows a move between two extremes : perfect fifths for Pythagorean tuning, and just major thirds for meantone tuning. Therefor, when initiating the meantone tempering with the aim for well tempering, one could aim for a good balance between fifths and major thirds impurities, instead of setting a just major third built on diminished fifths.

A possible balance, easy for auditory tuning, is to set a major third C – E, with a beating rate impurity equal to that of the four fifths that built it. This leads to a system with 4 equations ($q_{Note}$ and $p_{Note}$ : see tables 1 and 2) holding 4 variables : 4 unknown notes (C, G, D, E) :

$$q_C = q_G = q_D = q_A = p_C$$

**[5.3.2.2] The F and B notes**

Just as for the meantone, one can go on setting major thirds. But instead of being just, those should now hold an equal beating rate impurity. The continuation of equal beating rates of the major thirds on notes F and G means that we want :

$$q_C = q_G = q_D = q_A = p_C = p_F = p_G$$

This set of equations can be solved : it holds 6 equations with 6 variables : 6 unknown notes (F, C, G, D, E, B). Control of the solution shows that on top of the already desired impurity equalities, we obtain the same impurity also, for the fifth on E. We can therefor set:

$$q_C = q_G = q_D = q_A = p_C = p_F = p_G \quad = \quad q_E$$

This is a set of 7 equations with only 6 variables, but it can be solved indeed, because the seventh equation has been found redundant. The obtained note ratios are displayed below, and the obtained pitches and equal beating rates are on display in table 12 further on.

$$-q_{Note} = p_{Note} = \frac{A3}{113} = \frac{5F3}{451} = \frac{C4}{135} = \frac{G3}{101} = \frac{D4}{151} = \frac{E4}{169} = \frac{2B3}{253}$$

**[5.3.2.3] The altered notes of C–major**

The above obtained pitches for the F3 and B3 notes, embracing six diminished fifths, impose an average fifth ratio slightly exceeding 3/2, for the remaining six fifths. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{5 \times 253}{2 \times 451} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500396\ldots$$

The above ratio is slightly above perfection, but even closer to perfection than in paragraphs 5.2 and 5.3.1.2 above. The further definition of the altered notes can therefore be hold identical to the procedure applied in section 5.3.1.2.

The collection of solutions leads to the following scale, table 12 :

| | F3 | F#3 | G3 | G#3 | A3 | B*b*3 | B3 | C4 | C#4 | D4 | E*b*4 | E4 | F4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_{Note}$ | 175.61 | 184.71 | 196.64 | 207.80 | 220.00 | 234.02 | 246.28 | 262.83 | 277.07 | 293.98 | 311.90 | 329.03 | 351.22 |
| $q_{Note}$ | – 1.17 | 0.00 | – 1.95 | 0.39 | – 1.95 | 0.39 | 0.00 | – 1.95 | 0.00 | – 1.95 | 0.39 | – 1.95 | – 2.34 |
| $p_{Note}$ | 1.95 | 12.51 | 1.95 | 12.32 | 8.27 | 5.84 | 16.17 | 1.95 | 19.54 | 7.79 | 13.62 | 17.28 | 3.89 |

Table 12 : scale with quasi optimal beating rate equality for diatonic fifths and major thirds ; Δ(cent) = 0.42

As can be observed, the Well Tempered Meantone holds 5 fifths and 3 major thirds with a defined equal beating rate, equal to – 1.95 . . . bps. This equality of beatings rates should not be confused with the equal (cent–) impurities of the 12TET[2].

Figure 5 holds a horizontal axes obtained by deployment of the circle of fifths, that is opened at the G# (= Ab) note. It displays "unconventionally" the fifths and thirds impurities in bps., within a chromatic scale on C4 ; the courses within this figure are fairly irregular.

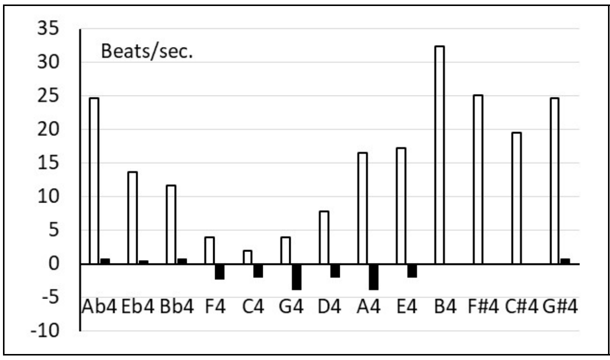


Figure 5 : Impurities of fifths (in black) and thirds

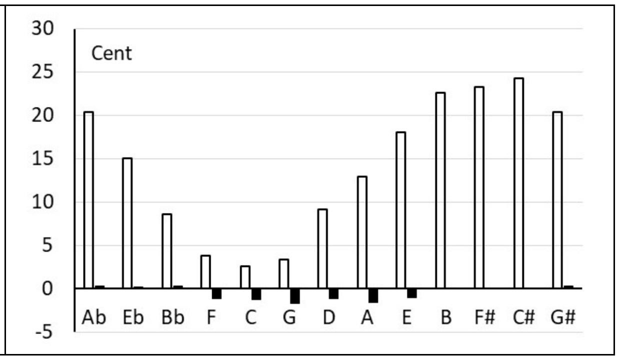


Figure 6 : Impurities of fifths (in black) and thirds

Figure 6 displays a “conventional” representation of the fifths and major thirds impurities in cents, and a rather normal course can be perceived, very comparable to those of most famous well temperaments. All fifths are purer than the purest major thirds, and the impurity of slightly augmented fifths on A*b* (G#), E*b*, B*b*, turns out to be insignificant.

Figure 7 “unconventionally” illustrates the fifths and thirds impurities in bps., within a chromatic scale on F3, the scale usually used for auditory tuning. The notes on the figure are given in an “unconventional” inverted sequence of fifths. The displayed course is remarkable and regular for the natural notes of C–major, and illustrates very well the obtained impurity equality of some fifths and major thirds.

Figure 8 illustrates in a similar way, the impurities expressed in cents. It has exactly the same course as fig. 6, but with an inversion and “phase shift” of the sequence of notes.

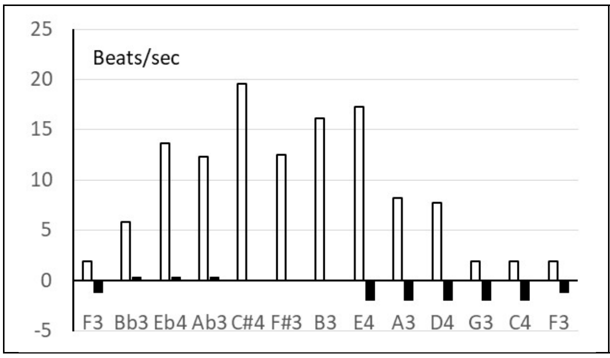


Figure 7 : Impurities of fifths (in black) and thirds

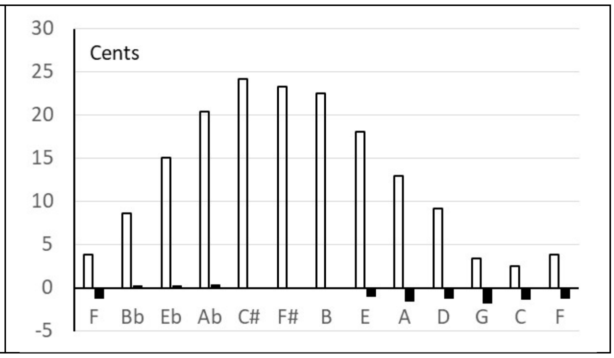


Figure 8 : Impurities of fifths (in black) and thirds

Figure 9 displays a clear and simple looking overview of the determined impurities distribution on the circle of fifths, whereby equality of markings on intervals signifies equality of interval beating rates :

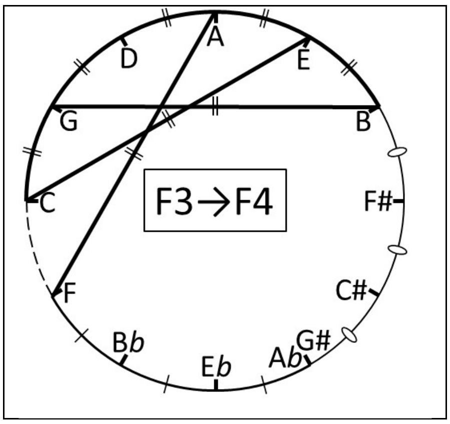


Figure 9 : Impurities distribution

### [5.3.3] The well tempered meantone ; tuning practice

#### [5.3.3.1] The well tempered meantone ; Chorton ("German organ diapason" ; A = 440)

The defined Well Tempered Meantone, table 7, paragraph 5.2.3, is still somewhat too complex for direct auditory application ; beating rates cannot be auditory measured up to two decimal digits precision. An alternative tuning instruction applied by Calvet holds fifths beating rates at – 2 , 0 , and 0.5 bps. Theoretical resulting pitches are displayed in table 13.

| | F3 | F#3 | G3 | G#3 | A3 | B*b*3 | B3 | C4 | C#4 | D4 | E*b*4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_{Note}$ | 175.67 | 184.69 | 196.67 | 207.77 | 220.00 | 234.06 | 246.25 | 262.89 | 277.03 | 294.00 | 311.91 | 329.00 |
| $q_{Note}$ | – 1.23 | 0.00 | – 2.00 | 0.50 | – 2.00 | 0.50 | 0.00 | – 2.00 | 0.00 | – 2.00 | 0.50 | – 2.00 |
| $p_{Note}$ | 1.66 | 12.79 | 1.67 | 12.69 | 8.13 | 5.71 | 16.39 | 1.56 | 20.19 | 7.50 | 13.78 | 17.19 |

Table 13 : Well Tempered Meantone ; possible tuning practice for "Chorton" (A = 440)

The diatonic major thirds beating rates become slightly slower, but remain mutually quasi equal. It appears to be a very good and practical applicable temperament.

#### [5.3.3.2] The well tempered meantone ; Kammerton ("German chamber diapason" ; A = 415)

The defined Well Tempered Meantone, table 7, paragraph 5.2.3, can also be worked out for a diapason of A = 415 ; see table 14. This can be set based on the instruction to set fifths at beating rates "slightly below – 2 ", and at 0 , and 1/3 bps.

| | F3 | F#3 | G3 | G#3 | A3 | B*b*3 | B3 | C4 | C#4 | D4 | E*b*4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_{Note}$ | 165.63 | 174.22 | 185.46 | 195.99 | 207.50 | 220.72 | 232.29 | 247.90 | 261.33 | 277.28 | 294.17 | 310.33 |
| $q_{Note}$ | – 1.10 | 0.00 | – 1.84 | 0.37 | – 1.84 | 0.37 | 0.00 | – 1.84 | 0.00 | – 1.84 | 0.37 | – 1.84 |
| $p_{Note}$ | 1.84 | 11.80 | 1.84 | 11.62 | 7.80 | 5.51 | 15.25 | 1.84 | 18.43 | 7.35 | 12.85 | 16.30 |

Table 14 : Well Tempered Meantone ; possible tuning practice for "Kammerton" (A = 415)

It appears to be a very good and practical applicable temperament.

### [5.4] The Well Tempered Meantone compared to Historic Temperaments

The above defined Well Tempered Meantone, table 12, paragraph 5.3.2 appears to probably be the best possible auditory tuneable well temperament.

The Δ(cent) difference of this Well Tempered Meantone, with the "optimal" well temperament table 10, paragraph 5.2 amounts to only 0.42 cents. The Δ(cent) difference of the Vallotti temperament with the same "optimal" well temperament amounts to 0.87 cents, the next best is Barca (acc. Devie) with 1.18 cents, and all other historical temperaments have still higher differences.

The Δ(cent) difference of the circular tempered meantone, table 11, paragraph 5.3.1, with the "optimal" well temperament amounts to 1.39 cents.

Observation :
Musical intuition might tend to accept higher beating rates for intervals on higher base notes.

This brings us back to the determination of notes based on cents, ratios or commas. The defined Well Tempered Meantone, table 12, paragraph 5.3.2 can also be compared to the cents–"optimal" well temperament defined in Appendix C.

Based on this cents–"optimal" temperament we obtain that the Δ(cent) difference with this Well Tempered Meantone amounts to only 0.66 cents. The Δ(cent) difference of the Vallotti temperament with the cents–"optimal" temperament amounts to 0.35 cents, the next best is Barca (acc. Devie) with 0.84 cents, and all other historical temperaments have still higher differences.

The Δ(cent) difference of the circular tempered meantone, table 6, paragraph 5.1.3, with the cents–"optimal" well temperament table 5 amounts to 1.58 cents.

**[5.5] "Das wohltemperirte Clavier"**

Well tempering becomes often related to Werckmeister, but it becomes very often related also to "Das wohltemperirte Clavier" (1722 ; 1740 – 42), a masterpiece of J. S. Bach.

Discussions on Bach temperaments can be controversial, due to the fact that J. S. Bach has not left any written instructions on how to tune a keyboard. No historic certainty at all exists on how he tuned his clavichord, although it is mentioned and generally accepted he was very skilled at it, and extremely rapid.

It may not be forgotten though, that Bach's musical education was based on the meantone, and that he probably only had some initial experiences with some well tempering during his visit to Buxtehude in Lubeck in 1705 (Kelletat, 19, p. 33, footnote 1). His well tempering might therefor have some links with the meantone, . . . question mark . . . ?

He was for sure very sensitive to musical affects: he enlightened the qualities of well temperaments by means of "Das wohltemperirte Clavier" (1722), but he also expressed horror in some parts of the "St. Matthew Passion" (1727), by intentional application of "forbidden" meantone keys (E, B, F#, C#, G#, E*b*) on meantone tuned instruments (Kelletat 1982, p. 20).

***Observation: the "St. Matthew Passion" (1727), where meantone can have a major impact on some musical affects, is POSTERIOR to "Das wohltemperirte Clavier" (1722) based on well tempering.***

Bach's clavichord tempering was discussed, sometimes indirectly, by Kirnberger (1771), in the letters of Kirnberger to Forkel (for example : Kelletat, 1960 ; 1981, pag. 41 ; 1982, pag. 34 footnote 12 ; . . . ), and by Forkel (1802), both arguing in favour of some kind of well temperament.

It was also discussed by Marpurg (1776, par. 228, p. 213), arguing in favour of the application of equal temperament (12TET) by Bach. It must be noticed Marpurg invokes Kirnberger, the latter should as a Bach student have been referring to Bach regarding this topic. Kirnberger, however, contradicting its pupil Marpurger, strongly denies this privately in letters to Forkel (Kelletat 1981, p. 42, footnote 18). The Marpurg assumption concerning Bach's temperament was copied by a countless number of authors in a countless number of publications, over two centuries, and even today.

A probably first doubt concerning the application of the 12TET by Bach was published by Bosanquet H. (1876, p. 28–30), and a breakthrough about those doubts probably came about through Kelletat (1960).

Kelletat assumes that the temperament that could have been applied by Bach could have been Kirnberger III, OR ANY OTHER SIMILAR ONE. Considering certain dates the later opinion is probably preferable : "or any other similar one". Indeed, "Das wohltemperirte Clavier" dates from 1722, Kirnberger was a student of Bach from 1739 to 1741, and Bach died in 1750 ; the Kirnberger I temperament dates from 1761, Kirnberger II from 1771, and Kirnberger III from 1779.

Since Kelletat many assumptions are published about possible "Bach–temperaments". To cite the most famous ones only, we can (chronologically) think of Kelletat (1966), Kellner (1977), Billeter (1979), Sparschuh (1999), Zapf (2001), Jobin (2005), Lehman (2005), Lindley (2006), Amiot (2008), and many others for sure are missing in this summary list (Calvet and Lehman for instance, publish longer lists).

Figure 10 illustrates the fifths impurities course in bps., of the Well Tempered Meantone defined in section [5.3], in a same notes sequence as on figures 7 and 8. This is also the sequence in which they can be assumed on a scrolled figure from J. S. Bach, on a score of “Das wohltemperirte Clavier” : an “inverted” fifths sequence, within the F3 – F4 range, – this is the notes range also used for tuning –. The scrolled part of figure 10 is copied from Amiot (2008).

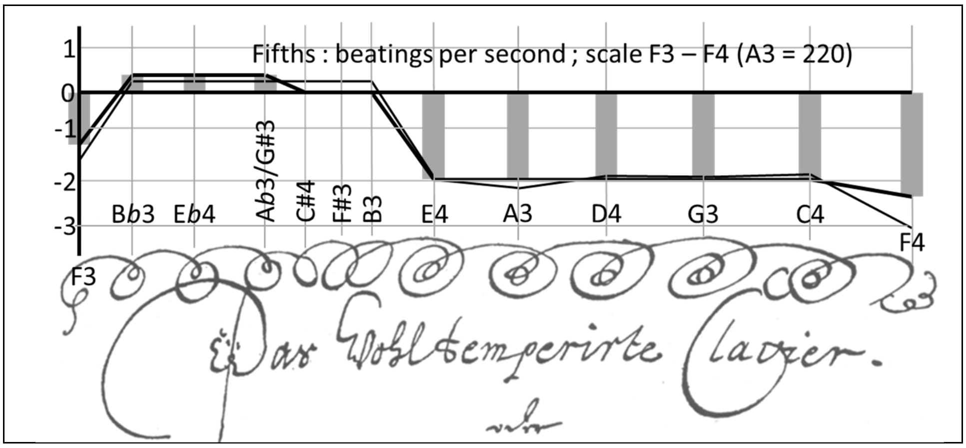


Figure 10 : “Well Tempered Meantone” fifths impurities, in bps. Fat line / grey bars (par. 5.3.2)
Slim line : “optimal” auditory tuning model (par. 5.2)

One can observe a striking parallelism between the note marks on the scrolls and the fifths beatings course, on the graph above the scrolls, possibly a closest fit, if compared with many other Bach–hypothesis. Remarkable also, is that this comparison is obtained following mathematical thoughts, where, on the other hand Bach was not interested in concerns about ratios or mathematics of temperaments, but the merely about harmony and purity (Forkel, p. 39[3]) :

> << As purposeful and reliable Bach’s style of teaching was in playing, so was it also in composition. He did not start with dry counterpoints that led to nothing, as was the case with other music teachers in his days ; still less did he hold up his students with calculations of the tonal relations which, in his opinion, did not belong to the composers, but the mere to theoreticians and instrument makers. >>

The exceptional dexterity and speed with which Bach could auditory tune a clavichord (Forkel, p. 17 ; Kelletat, 1981, p. 52 – 53) allows to assume that he very probably tuned the auditory way only.

If Bach should have tuned the above way indeed, than it could be said he solved in auditory way, a somewhat complex mathematic and auditory problem, of ***ease of auditory tuning, paired to well tempering, with an almost optimal diatonic C–major and with equal interval beating rates***.

## [6] Postamble

Famous historical temperaments, were defined based on interval beating rates instead of the more common application of interval ratios, or cent or comma deviations. It has become clear, that all those temperaments :

- are almost identical to their counterpart, commonly based on ratios, cents or comma’s.
- require only a diapason, sometimes maybe a metronome, for easy auditory tuning

It is not unreasonable to assume that Baroque temperaments were conceived, based on auditory observation by interpreting musicians and auditory tuners, and could therefore have been implicitly based on optimization of interval beating rates.

Post fact executed measurements by monochord at Baroque time, might at that time have lead to the commonly published definitions, based on cents, ratios or comma's, slightly differing from the actual installed one based on beating rates, ***because of unobserved very minor measuring errors* (*unobservable at that time* ! )**. Very minor measuring errors of a couple of cents are almost unavoidable, and the slight inharmonic structure and pitch sliding of the real physical vibrators of musical instruments also contribute to measuring uncertainties.

All above observations are valid in particular, for the Bach tuning according the Jobin proposal or the here derived beating rate based variant. The exceptional dexterity and speed with which Bach could auditory tune a clavichord (Forkel, p. 17 ; Kelletat, 1981, p. 52 – 53) allows to assume that he very probably tuned the auditory way only.

Based on all above arguments, it seems reasonable to assume that the spirals on the score of "Das wohltemperirte Clavier" may be associated to peculiar desired equalities of fifths and major thirds impurities concerning beatings and beating rates.

## [7] Conclusion

- Beating rate characteristics of historical temperaments are probably their determining factor. Beating rate characteristics should therefor deserve more attention in musicology, and calculation of temperaments, especially for those that were determined the auditory way, such as were most of those of the Baroque period.
- The proposed Well Tempered Meantone, section 5.3, offers a unique combination of ease of auditory tuning, paired to well tempering, with almost optimal diatonic C–major and with equal interval beating rates. This might be related to J. S. Bach, . . . question mark ?
- ***It is not the strict and exact mathematical equality*** of fifths and thirds impurities that prevails, . . . but that what prevails in this indeed, is the "comprehensive auditory equality judgment" of the interpreting musicians and auditory tuners, their "brainear" ("cervoreille" ; calvet 2020) observing sounds as comprehensive coherent groups, leading to virtual equal beating impurity of some fifths and major thirds.

ACKNOWLEDGMENT

I want to express my most sincere feelings of gratitude towards Amiot E., Calvet A., Jobin E., and Paintoux T, but also G. Baroin for the compilation of an interesting video on this subject.
Their open attitude allowed for quite intense exchange of ideas, that enabled further understanding and evolvement of insights in musical temperament and tuning matters, enabling the development of the ideas and concepts expressed in this paper.

Thanks to my daughter Hilde : she drew my attention to investigate on what musicians want (diatonic interval purity) and not on what might be someone's preferred musical temperament.

DEDICATION

This paper is dedicated to all classic musicians and auditory tuners.
Their sensitive musical ears offer to our world all the best of the most universal and most beautiful of all languages : ***MUSIC***.

Johan Broekaert
Master Electronics Engineering, Catholic University of Leuven (KULeuven), 1967
Nieuwelei, 52
2640 Mortsel
Belgium

0032 (0)3 455 09 85 Broekaert.devriendt@gmail.com

http://users.telenet.be/broekaert-devriendt/Index.html
https://home.deds.nl/~broekaert/Index.html

WORKS CITED

Amiot E. 2008 : "Discrete Fourier Transform and Bach's Good Temperament"
https://mtosmt.org/issues/mto.09.15.2/mto.09.15.2.amiot.html

Barbour J. M. 1951 : "Tuning and Temperament: A Historical Survey"

Bosanquet H. 1876 : "An elementary treatise on musical intervals and temperament"
https://en.xen.wiki/images/a/a7/Bosanquet_–_An_elementary_treatise_on_musical_intervals.pdf

Calvet A. 2020: "Le Clavier Bien Obtempéré"
http://www.andrecalvet.com/v3/index.php

De Bie J. 2001 : "Stemtoon en stemmingsstelsels", private edition, most data originate from Barbour J. 1951: "Tuning and Temperament: A Historical Survey"

de Zwolle H. A. (~1400 – 1466) : « Liber proportionum musicalium » (~1450).

Devie D. 1990 : "Le Tempérament Musical : philosophie, histoire, théorie et pratique"

Douwes C. 1699 : "Grondig onderzoek van de Toonen der Musyk",

Fritz B. 1756 : "Anweisung, wie man Claviere ... in allen zwölf Tönen gleich rein stimmen könne …"
https://gdz.sub.uni– goettingen.de/id/PPN630630391

Forkel J. 1802: Ueber Johann Sebastian Bachs Leben, Kunst und Kunstwerke.
https://www.digitale-sammlungen.de/en/view/bsb10528130?page=,1

Jedrzejewski F. 2002 : "*Mathématiques des systèmes acoustiques, Tempéraments et modèles contemporains.*" L'Harmattan, Paris, 2002.

Jobin E. 2005 : "BACH et le Clavier bien Tempéré" ;
https://www.clavecin– en– france.org/spip.php?article52.

Kelletat H. 1957 : "Ein Beitrag zur Orgelbeweging. Vom Klangerlebnis in nichtgleichswebenden Temperaturen"

Kelletat H. 1960 : "Zur musikalischen Temperatur"

Kelletat H. 1981 : "Zur musikalischen Temperatur ; Band I. Johann Sebastian Bach und seine Zeit"

Kelletat H. 1982 : "Zur musikalischen Temperatur ; Band II. Wiener Klassik"

Kelletat H. 1993 : "Zur musikalischen Temperatur ; Band III. Franz Schubert"

Kellner H. 1977 : "Eine Rekonstruktion der wohltemperierten Stimmung von Johann Sebastian Bach"

Kroesbergen W. 2013 : “18th Century Quotes on J.S. Bach’s Temperament” see : https://www.academia.edu/5210832/18th_Century_Quotations_Relating_to_J.S._Bach_s_Temperament

Lehman B. 2005 : “Bach’s extraordinary temperament: our Rosetta Stone – 1 ; – 2” ; Early Music

Marpurg F. 1776: “Versuch über die musikalische Temperatur“ http://www.deutschestextarchiv.de/book/view/marpurg_versuch_1776?p=5

Napier J. 1614 : “Mirifici logarithmorum canonis descriptio"

Norback J. 2002 : “A Passable and Good Temperament ; A New Methodology for Studying Tuning and Temperament in Organ Music”, https://core.ac.uk/download/pdf/16320601.pdf

Pitiscus B. 1603 : "Thesaurus mathematicus"

Railsback O. 1938 : “Scale temperament as applied to piano tuning” J. Acoust. Soc. Am. 9(3), p. 274 (1938)

Rameau J.– P. 1726 : Nouveau système de musique théorique

Rheticus G. 1542 : "De lateribus et angulis triangulorum (with Copernicus ; 1542)"

Salinas F. 1577 : “De musica libra septem” https://reader.digitale– sammlungen.de//resolve/display/bsb10138098.html

Sparschuh A. 1999 : “Stimm– Arithmetic des wohltemperierten Klaviers von J. S. Bach (TU Darmstadt)

Stevin S. ca. 1586 : “De Thiende” (The Tenth)

Stevin S. ca. 1605 : “Vande Spiegheling der Singconst” (Considerations on the art of singing) https://adcs.home.xs3all.nl/stevin/singconst/singconst.html

Werckmeister A. 1681 : “Orgelprobe”

Werckmeister A. 1686 : “Musicae Hodegus Curiosus” http://digitale.bibliothek.uni– halle.de/vd17/content/titleinfo/5173512

Werckmeister A. 1689 : “Orgelprobe” https://reader.digitale– sammlungen.de/de/fs1/object/display/bsb10527831_00007.html

Werckmeister A. 1691 : “Musicalische Temperatur” https://www.deutsche– digitale– bibliothek.de/item/VPHYMD3QYBZNQL2UAK3Q2O35GBSVAIEL

Zapf 2001: “Handling Down the Tradition: The Survival of Bach’s Fingering Technique in an Obscure Nineteenth–entury Clavier Tutor” (De Clavicordio, sept. 2001, pr. 39–44).

## Appendix A Beating rate recalculation of some historic temperaments

Table A1 displays the obtained note pitches and Δ(cent) between cent and beating rate versions.

| | Δ(cent) | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bendeler III 1690 | 0.66 | 175.01 | 185.00 | 196.22 | 208.12 | 220.00 | 233.35 | 246.66 | 262.51 | 277.50 | 293.33 | 311.13 | 330.00 |
| | | 175.08 | 185.17 | 196.36 | 208.32 | 220.00 | 233.45 | 246.89 | 262.63 | 277.76 | 293.33 | 311.26 | 330.00 |
| Kellner 1976 | 0.43 | 175.25 | 184.62 | 196.62 | 207.70 | 220.00 | 233.66 | 246.83 | 262.87 | 276.93 | 294.13 | 311.55 | 329.11 |
| | | 175.22 | 184.59 | 196.64 | 207.67 | 220.00 | 233.62 | 246.77 | 262.83 | 276.89 | 293.98 | 311.50 | 329.03 |
| Lehman 2005 | 0.52 | 175.40 | 185.21 | 196.44 | 208.12 | 220.00 | 233.61 | 246.94 | 262.51 | 277.81 | 294.00 | 311.83 | 329.26 |
| | | 175.45 | 185.19 | 196.42 | 208.14 | 220.00 | 233.68 | 246.92 | 262.40 | 277.78 | 293.85 | 311.83 | 329.23 |
| Mercadier 1788 | 0.43 | 175.21 | 184.89 | 196.44 | 207.65 | 220.00 | 233.61 | 246.73 | 262.51 | 277.10 | 294.00 | 311.48 | 329.26 |
| | | 175.22 | 184.92 | 196.43 | 207.67 | 220.00 | 233.63 | 246.76 | 262.44 | 277.09 | 293.86 | 311.50 | 329.21 |
| Neidhardt 1 1732 | 0.43 | 175.01 | 184.79 | 196.44 | 207.89 | 220.00 | 233.35 | 246.66 | 262.51 | 277.18 | 294.00 | 311.48 | 329.26 |
| | | 175.02 | 184.79 | 196.48 | 207.89 | 220.00 | 233.36 | 246.67 | 262.53 | 277.19 | 293.89 | 311.42 | 329.17 |
| Neidhardt 2 1732 | 0.42 | 175.21 | 185.21 | 196.44 | 207.89 | 220.00 | 233.87 | 247.22 | 262.51 | 277.50 | 294.00 | 311.83 | 329.63 |
| | | 175.20 | 185.21 | 196.42 | 207.88 | 220.00 | 233.86 | 247.21 | 262.41 | 277.43 | 293.85 | 311.81 | 329.61 |
| Neidhardt 3 1723 | 0.41 | 175.01 | 185.21 | 196.44 | 207.89 | 220.00 | 233.61 | 247.22 | 262.51 | 277.50 | 294.00 | 311.83 | 329.63 |
| | | 175.00 | 185.19 | 196.46 | 207.83 | 220.00 | 233.60 | 247.19 | 262.50 | 277.38 | 293.88 | 311.74 | 329.59 |
| Neidhardt 4 1732 | 0.44 | 175.01 | 184.79 | 196.66 | 207.65 | 220.00 | 233.61 | 246.66 | 262.51 | 277.18 | 294.33 | 311.48 | 328.88 |
| | | 174.98 | 184.73 | 196.64 | 207.62 | 220.00 | 233.57 | 246.58 | 262.46 | 277.10 | 294.15 | 311.43 | 328.78 |
| Sorge 1744 | 0.51 | 175.01 | 185.00 | 196.44 | 207.89 | 220.00 | 233.61 | 246.94 | 262.51 | 277.18 | 294.00 | 311.83 | 330.00 |
| | | 175.05 | 185.09 | 196.50 | 207.91 | 220.00 | 233.68 | 247.07 | 262.57 | 277.21 | 293.90 | 311.86 | 330.00 |
| Sorge 1758 | 0.85 | 175.01 | 185.21 | 196.44 | 208.12 | 220.00 | 233.61 | 247.22 | 262.51 | 277.50 | 294.00 | 311.83 | 329.63 |
| | | 175.00 | 185.19 | 196.46 | 207.83 | 220.00 | 233.60 | 247.19 | 262.50 | 277.38 | 293.88 | 311.74 | 329.59 |
| Stanhope 1806 | 0.58 | 175.27 | 184.75 | 197.18 | 207.85 | 220.00 | 233.70 | 246.48 | 262.91 | 277.13 | 294.55 | 311.77 | 328.64 |
| | | 175.24 | 184.70 | 197.14 | 207.79 | 220.00 | 233.65 | 246.43 | 262.86 | 277.06 | 294.29 | 311.69 | 328.57 |

| | Δ(cent) | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Vogel 1975 | 1.35 | 175.76 | 184.70 | 196.75 | 207.79 | 220.00 | 233.77 | 246.27 | 262.99 | 277.75 | 294.40 | 311.69 | 329.18 |
| | | 176.01 | 184.65 | 196.71 | 207.99 | 220.00 | 233.99 | 246.20 | 262.98 | 278.01 | 294.03 | 311.99 | 328.96 |
| Werckmeister VI 1691 | 0.48 | 175.18 | 185.27 | 196.69 | 207.62 | 220.00 | 234.02 | 247.50 | 262.77 | 276.83 | 292.77 | 312.03 | 330.00 |
| | | 175.23 | 185.34 | 196.84 | 207.63 | 220.00 | 234.02 | 247.50 | 262.84 | 276.85 | 292.95 | 312.03 | 330.00 |

Table A1 : Comparison of temperament pitches : the lower rows are the beating rate calculated versions.

Probably many more temperaments could be redefined or recalculated (see also : Calvet, Jedrzejewski F. 2002).

Table A2 displays the applied comma divisions (q/n and/or p/n) and the obtained beating rates.

| | **Auditory tuning instructions** (calculation instructions) |
|---|---|
| ***Important note :*** $q_{Note}$ that are not listed below, have to be set to the value **<< $q_{Note}$=0 >>** | |
| Bendeler III 1690 | $q_C=q_G=q_E=q_{G\#}=1/4$ ; beating(q/4) = – 2.43 |
| Kellner 1976 | $q_C=q_G=q_D=q_A=q_B=1/5$ ; beating(q/5) = – 1.94 |
| Lehman 2005 | $q_F=q_C=q_G=q_D=q_A=2q_{C\#}=2q_{G\#}=2q_{Eb}=-2q_{Bb}=1/6$ ; beating(q/6) = – 1.55 |
| Mercadier 1788 | $3q_E=3q_B=3q_{F\#}=3q_{C\#}=6q_F=8q_C=8q_G=8q_D=8q_A=1/2$ ; beating(q/6) = – 1.58 |
| Neidhardt 1 1732 | $q_C=q_G=q_D=q_A=2q_E=2q_B=2q_{G\#}=2q_{Eb}=1/6$ ; beating(q/6) = – 1.66 |
| Neidhardt 2 1732 | $q_C=q_G=q_D=2q_F=2q_A=2q_B=2q_{F\#}=2q_{C\#}=2q_{Bb}=1/12$ ; beating(q/6) = – 1.56 |
| Neidhardt 3 1723 | $q_C=q_G=q_D=2q_A=2q_B=2q_{F\#}=2q_{C\#}=2q_{Eb}=2q_{Bb}=1/12$ ; beating(q/6) = – 1.63 |
| Neidhardt 4 1732 | $2q_D=2q_A=3q_G=6q_C=6q_B=6q_{Bb}=6q_{C\#}=1/2$ ; beating(q/6) = – 1.63 |
| Sorge 1744 | $q_C=q_G=q_D=q_E=2q_B=2q_{F\#}=2q_{Eb}=2q_{Bb}=1/6$ ; beating(q/12) = – 1.71 |
| Sorge 1758 | $q_C=q_G=q_D=2q_A=2q_B=2q_{F\#}=2q_{C\#}=2q_{Eb}=2q_{Bb}=1/6$ ; beating(q/12) = – 1.63 |
| Stanhope 1806 | **Third** : $p_C=0$ ; and **fifths** : $q_G=q_D=q_A$ ; $q_B=q_{Eb}$ ; leads to $q_G$ =– 2.86 ; $q_B$= – 0.47 |
| Vogel 1975 | $q_F=q_C=q_G=q_A=q_D=q_E=-q_{F\#}=-q_B=1/5$ ; beating(q/5)=– 2.08 |
| Werckmeister VI 1691 | $q_G=2q_{F\#}=4q_C=4q_B=4q_{Bb}=-4q_D=-4q_{G\#}=4/7$ l beating(q/7)=– 1.16 |

Table A2 : Tuning instructions and beatings of redefined historical temperaments

## Appendix B Optimisation of diatonic characteristics

### Appendix B1 Optimal C–major Diatonic Purity, for a lowest BEATING RATE impurity, for fifths on C4, G3, D4, A3, E4 and major thirds on F3, C4, G3.

This impurity measure corresponds to the following sum of squared beating rates :

$$\sum(q_{Note}^2 + p_{Note}^2) = q_F^2 + q_C^2 + q_G^2 + q_D^2 + q_A^2 + q_E^2 \qquad + \qquad p_F^2 + p_C^2 + p_G^2$$

This sum can be worked out in function of the notes :

$$34F_3^2 + 38C_4^2 + 50G_3^2 + 13D_4^2 + 41A_3^2 + 29E_4^2 + 32B_3^2$$
$$-12F_3C_4 - 40F_3A_3 - 24C_4G_3 - 40C_4E_4 - 12G_3D_4 - 40G_3B_3 - 24D_4A_3 - 12A_3E_4 - 24E_3B_4$$

The partial derivatives to the notes, set to zero and after simplification of coefficients, lead to the set of linear equations, table B1 :

| | F3 | C4 | G3 | D4 | E4 | B3 | = | A3 |
|---|---|---|---|---|---|---|---|---|
| ∂/∂F3 : | 17 | –3 | 0 | 0 | 0 | 0 | = | 10 |
| ∂/∂C4 : | –3 | 19 | –6 | 0 | –10 | 0 | = | 0 |
| ∂/∂G3 : | 0 | –6 | 25 | –3 | 0 | –10 | = | 0 |
| ∂/∂D4 : | 0 | 0 | –6 | 13 | 0 | 0 | = | 12 |
| ∂/∂E4 : | 0 | –20 | 0 | 0 | 29 | –12 | = | 6 |
| ∂/∂B3 : | 0 | 0 | –5 | 0 | –3 | 8 | = | 0 |

Table B1 : Calculation of the optimal diatonic C–major impurity of fifths and major thirds

The five remaining notes, F#3, C#4, G#3, E*b*4, B*b*3, and their beatings, are obtained by setting an equal beating for the six involved fifths :

$$q_{Note} = q_B = q_{F\#} = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb}$$

The obtained temperament is displayed in table B2 :

| | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pitches | 175.86 | 184.98 | 197.08 | 208.24 | 220.00 | 234.40 | 246.57 | 263.19 | 277.58 | 294.04 | 312.47 | 329.06 |
| Fifth beatings | – 1.19 | 0.22 | – 3.16 | 0.22 | – 1.89 | 0.22 | 0.22 | – 1.26 | 0.22 | – 2.11 | 0.22 | – 0.89 |
| Major thirds beatings | 0.72 | 12.71 | 0.89 | 11.56 | 10.32 | 4.12 | 17.02 | 0.28 | 18.96 | 9.67 | 14.29 | 20.63 |

Table B2 : Scale with minimal beating rate of C–major diatonic fifths and major thirds l Δ(cent) = 0.38

**Appendix B2 Optimal C–major Diatonic Purity, for a lowest CENT impurity, for fifths on C4, G3, D4, A3, E4 and major thirds on F3, C4, G3.**

The proposed purity calculation in cents, on the F3 – F4 scale, is proportional to the sum below :

$$\left[\log_2\left(\frac{2C_4}{3F_3}\right)\right]^2 + \left[\log_2\left(\frac{4G_3}{3C_4}\right)\right]^2 + \left[\log_2\left(\frac{2D_4}{3G_3}\right)\right]^2 + \left[\log_2\left(\frac{4A_3}{3D_4}\right)\right]^2 + \left[\log_2\left(\frac{2E_4}{3A_3}\right)\right]^2 + \left[\log_2\left(\frac{4B_3}{3E_4}\right)\right]^2$$
$$+ \left[\log_2\left(\frac{4A_3}{5F_3}\right)\right]^2 + \left[\log_2\left(\frac{4E_4}{5C_4}\right)\right]^2 + \left[\log_2\left(\frac{4B_3}{5G_3}\right)\right]^2$$

The five remaining notes, F#3, C#4, G#3, E*b*4, B*b*3 are obtained by setting an equal impurity for the six involved fifths ; the major thirds do not have to be optimized, those have rapid beating rates anyhow.

| | log₂(F3) | log₂(C4) | log₂(G3) | log₂(D4) | log₂(E4) | log₂(B3) | = | A3 |
|---|---|---|---|---|---|---|---|---|
| ∂/∂F3 : | 2 | – 1 | 0 | 0 | 0 | 0 | = | $\log_2 A3+3+\log_2 3-\log_2 5$ |
| ∂/∂C4 : | – 1 | 3 | – 1 | 0 | – 1 | 0 | = | $3-\log_2 5$ |
| ∂/∂G3 : | 0 | – 1 | 3 | – 1 | 0 | – 1 | = | $1-\log_2 5$ |
| ∂/∂D4 : | 0 | 0 | – 1 | 2 | 0 | 0 | = | $\log_2 A3+1$ |
| ∂/∂E4 : | 0 | – 1 | 0 | 0 | 3 | – 1 | = | $\log_2 A3-1+\log_2 5$ |
| ∂/∂B3 : | 0 | 0 | – 1 | 0 | – 1 | 2 | = | $-4+\log_2 3+\log_2 5$ |

Table B3 : Calculation of the optimal diatonic C–major impurity of fifths and major thirds, for impurities expressed in cents

The obtained note pitches and interval beating rates are on display in table B4 :

| | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pitches | 175.74 | 184.98 | 196.99 | 208.18 | 220.00 | 234.28 | 246.60 | 263.23 | 277.52 | 294.41 | 312.32 | 329.28 |
| Fifth beatings | – 0.77 | 0.10 | – 2.16 | 0.11 | – 1.45 | 0.12 | 0.13 | – 1.73 | 0.15 | – 3.22 | 0.17 | – 1.44 |
| Major thirds beatings | 1.29 | 12.23 | 1.44 | 12.03 | 10.08 | 6.22 | 16.30 | 0.96 | 18.34 | 7.81 | 14.31 | 19.03 |

Table B4 : Scale of the optimal diatonic C–major impurity of fifths and major thirds, for impurities calculated in cents

**Appendix B3 Observation**

Both obtained "optimal impurity" temperaments are very similar, as expected : Δ(cent) = 0.38.

Both temperaments have C–major diatonic fifths beating rates within their own scale, that are quite dissimilar : from – 0.77 to – 3.22 bps. This is not comfortable for auditory tuning.

**Appendix C Equal cent–impurity temperament**

Unlike with beat rate calculations, it is possible to define diatonic C–major fifths and major thirds, ALL of those holding equal impurities, for impurities calculated in cent.

Four equal impurity fifths must lead to an equal impurity major third. Setting "$k$" as impurity factor, we can therefor set:

$$\left(k\frac{3}{2}\right)^4 \cdot 2^n = \frac{1}{k}\cdot\frac{5}{4} \quad \text{with} \quad n = \text{an appropriate integer}$$

$$\text{This leads to :} \qquad k^5 = \frac{5\cdot 2^4}{3^4} \quad or \quad k = 0.99751858$$

The six remaining fifths, will hold an equal impurity "$m$", that has to comply with:

$$\left(m\frac{3}{2}\right)^6 \cdot \left(k\frac{3}{2}\right)^6 \cdot 2^p = 1 \quad \text{with} \quad p = \text{an appropriate integer, leading to} \quad m = \sqrt[5]{\frac{\sqrt[6]{2^{71}}}{5\cdot 3^6}} = 1.000226$$

The obtained note pitches become :

| | F3 | F#3 | G3 | G#3 | A3 | Bb3 | B3 | C4 | C#4 | D4 | Eb4 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pitches | 175.56 | 184.75 | 196.53 | 207.93 | 220.00 | 234.03 | 246.27 | 262.69 | 277.18 | 294.06 | 311.97 | 329.18 |
| Fifth impurities | – 4.30 | 0.39 | – 4.30 | 0.39 | – 4.30 | 0.39 | 0.39 | – 4.30 | 0.39 | – 4.30 | 0.39 | – 4.30 |
| Major thirds impurities | 4.30 | 23.07 | 4.30 | 18.38 | 13.69 | 8.99 | 23.07 | 4.30 | 23.07 | 8.99 | 13.69 | 18.38 |

Table C1 : Scale of the optimal diatonic C–major impurity of fifths and major thirds, for equal impurities expressed in cents

[1] Original German definition :
"Wohltemperierung heißt mathematisch-akustische und praktisch-musikalischen Einrichtung von Tonmaterial innerhalb der zwölfstufigen Oktavskala zum einwandfreien Gebrauch in allen Tonarten auf der Grundlage des natürlich-harmonischen Systems mit Bestreben möglichster Reinerhaltung der diatonische Intervalle.
Sie tritt auf als proportionsgebundene, sparsam temperierende Lockerung und Dehnung des mitteltönigen Systems, als ungleichschwebende Semitonik und als gleichschwebende Temperatur."

[2] Misunderstanding might be possible in the German language : the here obtained equal beating corresponds to "Gleichschwebend", whereby on the other hand the 12TET is named by "Gleichschwebende Temperatur". The same confusion is possible too in the Dutch language.

[3] Original German text :
So zweckmäßig und sicher Bachs Lehrart im Spielen war, so war sie es auch in der Composition. Den Anfang machte er nicht mit trockenen, zu nichts führenden Contrapuncten, wie es zu seiner Zeit von andern Musiklehrern geschah; noch weniger hielt er seine Schüler mit Berechnungen der Tonverhältnisse auf, die nach seiner Meynung nicht für den Componisten, sondern für den bloßen Theoretiker und Instrumentenmacher gehörten.