The Well Tempered Meantone ___ brief

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Abstract :

A hypothetical virtual reconstruction of a quest for a well tempered auditory musical keyboard tuning derived from the meantone is worked out, based on Werckmeister's concepts about well tempering, and musical insights and auditory music keyboard tuning practices at Baroque times, but also nowadays. The elaborated Well Tempered Meantone turns out to fit best with an optimum, based on presumed auditory tuning quality criteria. A number of characteristics that can be related to J. S. Bach, are discussed.

Keywords

Baroque ; well temperament ; meantone ; interval ; comma ; beating ; harmonic ; ratio ; cent ; Bach

[1] Methodology

The reflections on temperaments in this paper, are based on tuning procedures applied by auditory musical keyboard tuners. The professional tuning of a musical keyboard at Baroque time and still today can be done a pure auditory way, by evaluation of interval beating rates, not using any measuring tool at all, except for a diapason for calibration, and sometimes a metronome to enhance the tuning precision. Application of a monochord was usually not required.

A current procedure often relies on an initial tuning of all the notes within the chromatic scale from F3 to F4 (Calvet, 2020). The notes within this scale have rather low pitches and contain many harmonics, facilitating the tuning because of clear and slow beating rates of intervals. Moreover, the major third on C will have the best ratio of all thirds, if all C–major diatonic major thirds have equal beating rate, because this major third has the highest pitch due to its high position on the F3 – F4 scale. The F3 – F4 scale also stands at a critical transition of string characteristics on a piano. This initial F3 – F4 scale setting, and further tuning mainly relies on the observation of interval beatings of fifths and major thirds.

Beating rates within the F3 – F4 scale can be calculated with the equations tables 1 and 2. The q_{Note} and p_{Note} symbols stand for the beating rates of fifths and major thirds. The formulas for q_C and p_C were applied by A. Kellner (1977). Formulas for minor thirds also, can be established.

$q_F = 2C4 - 3F3$	$q_C = 4G3 - 3C4$	$q_G = 2D4 - 3G3$	$q_D = 4A3 - 3D4$
$q_A = 2E4 - 3A3$	$q_E = 4B3 - 3E4$	$q_B = 4F\#3 - 3B3$	$q_{F\#} = 2C\#4 - 3F\#3$
$q_{C\#} = 4G\#3 - 3C\#4$	$q_{G\#} = 2Eb4 - 3G\#3$	$q_{Eb} = 4Bb3 - 3Eb4$	$q_{Bb} = 4F3 - 3Bb3$

Table 1 : calculation of fifths beating rate within the F3 – F4 scale

$p_F = 4A3 - 5F3$	$p_C = 4E4 - 5C4$	$p_G = 4B3 - 5G3$	$p_D = 8F\#3 - 5D4$
$p_A = 4C\#4 - 5A3$	$p_E = 8G\#3 - 5E4$	$p_B = 4Eb4 - 5B3$	$p_{F\#} = 4Bb3 - 5F\#3$
$p_{C\#} = 8F3 - 5C\#4$	$p_{G\#} = 2C4 - 5G\#3$	$p_{Eb} = 8G3 - 5Eb4$	$p_{Bb} = 4D4 - 5Bb3$

Table 2 : calculation of major thirds beating rate within the scale F3 – F4

[2] Comparison of Temperaments

Some discussed temperaments will have to be evaluated. Part of an evaluation can consist of the comparison of temperaments.

Although this paper relies on the interpretation of interval beating rates, rather than interval ratios, it is not adequate to use measurements in beatings per second (bps) as such for the comparison of temperaments : the beating rate is proportional to note pitch indeed, and therefor depends on the note pitch and the used diapason. Hence, a relative measure is preferable instead of an absolute measure in bps. An adequate and most common relative interval measure is the cent.

Characteristics of any temperament are unambiguously defined by either : the note pitches that depend on the set diapason, or relative measures such as semi–tones, fourths, fifths or sevenths (= 11 semi–tones). Fifths characteristics are most commonly applied ; just think of the circle of fifths.

Therefor, a global and objective comparison of temperaments can be based, for example, on a measure in cents of mutual fifths differences, by applying the formula below :

$$\Delta(\text{cent}) = \sqrt{\frac{\sum_{F,C,G,D,A,E,B,F\#,C\#,G\#,Eb,Bb}\left(fifth_{Note;temperament\_1} - fifth_{Note;temperament\_2}\right)^2}{12}}$$

[3] The Meantone

Just major thirds, ratio 5/4, are in general appreciated and desired by musicians and their audience. The quality of major thirds can determine the differing colours, characters or affects of differing temperaments. Hence, much importance is paid to the just major third. Just major thirds, ratio 5/4 (= 1.25), differ from the rather sharp ones of the Pythagorean tuning, having a ratio 81/64 (= 1.265625). The difference is the syntonic comma, with ratio 81/80. The twelve tone equal temperament (12TET) major thirds also, are quite sharp, with a ratio $2^{4/12}$ (= 1.2599 . . .). Early considerations on the importance of the just major thirds, can, among others, be attributed to Ptolemaeus (c. 100 − c. 170).

The just major thirds became widely introduced and accepted because of the meantone temperament. This temperament was described in 1523 by P. Aaron and by Salinas (1577) according Zarlino. This meantone is based on a just major third on C that is divided in four equal halftones. This temperament is therefore called the “quarter (syntonic) comma meantone”. All other notes emanate from just major thirds on the notes E*b*, B*b*, F, (C), G, D, A, E.

Please notice : ALL allowed meantone keys* (B*b*, F, C, G, D, A,) *offer EXCELLENT and IDENTICAL HARMONY, the only difference between the allowed keys consists of a tonal height difference only.
Therefore, modulation is musically completely free for meantone tuned instruments, as long as it remains within the allowed keys.
The meantone was probably the “dominating” temperament for Baroque music.

For the meantone also, one may wonder whether the C − E major third should be divided based on ratios, cents or commas, as is commonly assumed, presumably by means of a monochord or measuring instrument, or whether it should somehow have been based on beating rates ?

It is clear that the equal division in four half tones of the C − E major third, corresponds with four equal fifths (A − E, D − A, G − D, C − G) on which this major third is built. Therefore, a hypothetical alternative division might consist of building the just major third C − E, based on four fifths with equal beating rate (A − E, D − A, G − D, C − G), followed by the further installation of seven

additional and desired just major thirds. This determination of notes also has the advantage it contains a direct link with the A note when initiating the tuning, but also with the C note ; the latter imports if the C note is chosen as diapason, as was or is often done.
The initial note pitches can thus be obtained by solving following equations (see tables 1 and 2) :

A just major third on C : $p_C = 0$
and equal beating fifths on C, G, D, A : $q_C = q_G = q_D = q_A = q_{Note}$

The remaining notes are defined, by setting just major thirds on the notes E*b*, B*b*, F, G, D, A, E :

$$p_{Eb} = p_{Bb} = p_F = p_G = p_D = p_A = p_E = 0$$

Table 3 displays the obtained pitches, pitch differences (Δ–pitch, ***in cents***), beating rates and the overall fifths differences Δ(cent) (according the formula paragraph 2) with the classical version :

Meantone		F3	F#3	G3	G#3	A3	Bb3	B3	C4	C#4	D4	Eb4	E4
"classic"	Pitches	176.00	183.90	196.77	205.61	220.00	235.40	245.97	263.18	275.00	294.25	314.84	328.98
	fifth beats	– 1.64	– 1.71	– 1.83	12.85	– 2.05	– 2.19	– 2.29	– 2.45	– 2.56	– 2.74	– 2.93	– 3.06
Equal beating	Pitches	176.00	183.79	196.78	205.56	220.00	235.26	245.98	263.12	275.00	294.07	314.85	328.89
	Δ–pitch	0.00	– 1.03	0.09	– 0.43	0.00	– 1.03	0.09	– 0.43	0.00	– 1.03	0.09	– 0.43
	fifth beats	– 1.77	– 1.38	– 2.21	13.03	– 2.21	– 1.77	– 2.76	– 2.21	– 2.76	– 2.21	– 3.54	– 2.76

Table 3 : Comparison of the "classic" meantone with the auditory tuned one (for A4 = 440) ; Δ(cent) = 0.43 ; Δ-pitch ***in cents***

This proposed alternative auditory tuning could be installed at much ease by any Baroque tuner, not using any measuring tool at all.

Precise tuning of the "classic" division of the just C − E third was only possible by means of a monochord, but this goes at par with quite intensive labour : for a monochord of length of 1000 mm., tuned on the note C ; the movable bridge must be set at C# = 935.73 mm, D = 893.33 mm, Eb = 835.90 mm, and E = 800.00 mm, and these settings are not connected yet with the A note. A deviation of only 1 mm corresponds to a pitch deviation of 2.16 . . . cents already. This is more than twice the maximum differences displayed at table 3 (1.03 cent). Doubts can therefore be expressed on the effective installation during the Baroque period, of the nowadays commonly published pitches of the meantone temperament, based on ratio calculations.

There is in fact no single practical reason not to tune by the ear. The differences between both versions are practically and auditory not distinguishable.

[4] **Well Temperaments** (circulating temperaments)

Well temperaments strive for best interval purities for the diatonic C–major, and acceptable purity for the other tonalities. A musical definition of well temperaments, based on the Werckmeister criteria, was elaborated by H. Kelletat (1960 ; 1981, p. 9) :

> << Well temperament means a mathematical–acoustic and musical–practical organization of the tone system within the twelve steps of an octave, so that impeccable performance in all tonalities is enabled, based on the extended just intonation (natural–harmonic tone system), while striving to keep the diatonic intervals as pure as possible.
> This temperament acts, while tied to given pitch ratios, as a thriftily tempered smoothing and extension of the meantone, as unequally beating half tones and as equal (equally beating) temperament. >> (see endnote for original German text [1])

There has been a general and very active quest for well temperaments at Baroque time. Reason for this quest was that the meantone does not allow for acceptable musical modulation in all keys, because of the "wolf fifth" (usually on G#) and the associated four "harsh" major thirds.

Werckmeister was, among others, at the origin of the well temperament concept, and his well tempered Werckmeister III (1691) temperament became famous. A well temperament might have been requested for the reconstruction of the organ of the Lucca Cathedral (Italy), in 1473 (Devie1990, p. 55). Werckmeister is probably also the first to have used the "wohltemperiert" (well tempered) term in writing :

- (Werckmeister, 1681 ; Norback, 2002, p 18, fig. 2, title page) : "…ein Clavier *wohl zu temperiren* und zo stimmen sei / …"
- (Werckmeister, 1686) :
  - Chapter 30, page 118 (erratically marked as 108) : "…/ wenn unser Clavier *wol temperiret* ist / …"
  - Page 120, 16–th rule : "…Wenn wir hingegen ein *wohl temperirtes* Clavier haben / …"
- (Werckmeister, 1691) :
  - Title page : "…/ und dergleichen *wol temperirt*…"
  - Chapter 22, page 61, 7–th rule : "…/ wenn nun ein Clavier *wohl temperiret* ist / …"
- (Werckmeister, 1698, p. 7) : "… / und wäre nicht *wohl temperiret* oder …"

Nowadays again, well temperaments have become a hot musical topic. It is quite probable that the publications of H. Kelletat (1956, 1960, 1981, 1982) are at the origin of the present interest. Kelletat in turn refers to Bosanquet (1876). A. Calvet (2020) also, in a more recent publication, such as H. Kelletat, deploys considerations in width and in depth on the musical temperament and tuning topics, supported by historical aspects and profound explanation on how and why musical temperaments, intervals and interval beating rates have specific important characteristics. Calvet has treated in particular and depth also the aspect of musical interval beating, and the beating rates of many temperaments are well documented. He is also discussing the importance of required interval readjustments during the practice of piano tuning, because of inharmonicity of strings, or corrections for better distribution of beatings, and those corrections might therefore be due to desired auditory beating rate improvements leading to deviations from published pitches. Jobin (2005) also, mentions the necessity of interval readjustments.

[4.1] Mathematical definition of an optimal auditory well temperament

One might wonder for an optimal well temperament, holding an optimal C–major diatonic purity.

An objective criterion is required for the mathematical elaboration of such an optimum. But even so, it is not probable that such a criterion could be imposed musically, and musicians will probably not agree upon one : it is indeed sufficient to overlook the vast collection of well temperaments to get convinced about the musical subjectivity of the topic. Nevertheless, having some optimum can be useful and so we make an attempt to define one.

Since the auditory tuning is based on setting C–major diatonic fifths and major thirds, one can think of optimising their impurities. Mathematical determination of an optimal temperament with minimal auditory beating for the six fifths and three major thirds of the diatonic C–major leads to rather high differences in beating rates of fifths, uncomfortable for auditory tuning : those beating rates vary from – 0.77 to – 3.22 bps.

In auditory tuning practice it is not possible to set a minimum beating rate, without application of high precision measuring tools. Those tools are required for high precision measurement of the beating rates of six fifths and three major thirds, their sum, and their minimum.

It is probably musically better, and also easier for auditory tuning, to estimate the equality of slow beating rates, rather than their minimum sum ; no tool at all is required to make a good estimation of beating rate equalities. But exact equality is probably not possible : we have nine equations (beating rates of six fifths and three major thirds) for six variables (F, C, G, D, E, B). We can therefor only strive for a best possible equality. A best possible equality corresponds to a minimum deviation from an average value "M".

Beatings are normally negative on fifths (too small), and positive on thirds (too large). Normally the absolute average beating rate "M" therefore is :

$$M = \frac{-q_F - q_C - q_G - q_D - q_A - q_E + p_F + p_C + p_G}{9}$$

Taking the signs into account, the deviations from the mean beating are :

Fifths : $\Delta_{Fi;Note} = -q_{Note} - M$ Major Thirds : $\Delta_{T;Note} = p_{Note} - M$

The appropriate sum of the squares of deviations becomes :

$$\sum \Delta^2_{Fi\ and\ T;Note} = \Delta^2_{Fi;F} + \Delta^2_{Fi;C} + \Delta^2_{Fi;G} + \Delta^2_{Fi;D} + \Delta^2_{Fi;A} + \Delta^2_{Fi;E} \quad + \quad \Delta^2_{T;F} + \Delta^2_{T;C} + \Delta^2_{T;G}$$

the elaboration of this sum in function of the notes leads to :

$$\begin{aligned} 81 \times \sum \Delta^2_{Fi\ and\ T;Note} = {} & 2718F_3^2 + 2934C_4^2 + 3726G_3^2 + 1044D_4^2 + 3240A_3^2 + 2124E_4^2 + 2592B_3^2 \\ & -1116F_3C_4 - 216F_3G_3 + 36F_3D_4 - 3132F_3A_3 + 180F_3E_4 \\ & -2376C_4G_3 + 72C_4D_4 + 216C_4A_3 - 2880C_4E_4 \\ & -864G_3D_4 + 324G_3A_3 + 540G_3E_4 - 3240G_3B_3 \\ & -1998D_4A_3 - 90D_4E_4 \quad - 1242A_3E_4 \quad - 1944E_4B_3 \end{aligned}$$

Following an advice from E. Amiot, the minimum of this sum can be determined by calculation of the partial derivatives to the variables, set to equal zero. Table 4 displays the simplified coefficients of equations obtained from the partial derivatives to the notes :

	F3	C4	G3	D4	E4	B3	=	A3
∂/∂F3 :	151	– 31	– 6	1	5	0	=	87
∂/∂C4 :	– 31	163	– 66	2	– 80	0	=	– 6
∂/∂G3 :	– 2	– 22	69	– 8	5	– 30	=	– 3
∂/∂D4 :	2	4	– 48	116	– 5	0	=	111
∂/∂E4 :	10	– 160	30	– 5	236	– 108	=	69
∂/∂B3 :	0	0	– 5	0	– 3	8	=	0

Table 4 : Calculation of diatonic notes for C – major

The obtained B and F pitches (see further table 5) impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{246.22}{175.67} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500545\ldots$$

The above ratio is slightly above perfection, but so little that it still can be acceptable within a well temperament. Further creation of an “optimal” well temperament therefor, only allows for even distribution of this minute obligate fifths augmentation over the six remaining fifths.

$$q_{Note} = q_B = q_{F\#} = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb}$$

The collection of solutions leads to the scale, table 5 :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.67	184.73	196.60	207.98	220.00	234.14	246.22	262.75	277.22	293.96	312.10	328.93
q_{Note}	– 1.52	0.26	– 1.89	0.26	– 2.15	0.26	0.26	– 1.83	0.26	– 1.87	0.26	– 1.89
p_{Note}	1.65	12.91	1.89	11.07	8.90	5.13	17.30	1.97	19.23	8.06	12.30	19.23

Table 5 : scale with optimal beating rate equality of the of diatonic fifths and major thirds within C major

The average beating rate for C–major diatonic fifths and major thirds between F3 and F4, is – 1.85 . . . bps., with minor deviations only from this value (≤ 0,33).

[5] Creation of a Well Tempered Meantone

Both, the Pythagorean system with 11 perfect fifths as well as the meantone system with 8 just major thirds, hold an undesired wolf fifth. Such as the equal temperament was derived from the Pythagorean system, one can strive to derive a well temperament from the meantone (cfr. the well temperament definition, section 4). Tempering of the meantone wolf fifth has lead to a number of historic meantone variants, among those some holding less just major thirds such as Rameau (4 MT), Marpurg, (4 MT), Legros (3 and 2 MT), d’Alembert (1 MT), de Béthisy (1 MT), . . . or others with differing division of the syntonic comma, such as Salinas (1/3 comma), Zarlino (2/7 comma), Sauveur (1/5 comma), Romieu (1/6 and 1/7 comma), . . .

[5.1] A circulating tempered meantone holding less just major thirds (auditory tuneable)

[5.1.1] The natural diatonic notes of C–Major

Inspired by E. Jobin (2005), the tuning can be initiated, installing three just major thirds within the diatonic C–major ; one on C, and those on F and G to follow ; Jobin defined two just major thirds. All natural diatonic notes have thus become defined. Their pitches are equal to those that are already on display in table 3 (the equal beating meantone version).

[5.1.2] The altered notes of C–Major

The obtained B and F pitches impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{245.98}{176.00} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.501258\ldots$$

The above ratio is slightly above perfection, but so little that it can be acceptable still, within a well temperament. Further creation of a well temperament therefor, does no more allow for additional major thirds comparable to those already defined, while this leads to a further and noticeable enlargement of the remaining fifths.

For ease of auditory tuning, but in line with classic Pythagorean tuning, the notes F#, C# and G# can become defined instead, by perfect fifths on B, F# and C# ; those perfect fifths have an extremely minor difference only, with the above determined mean ratio (1.501258 . . .) :

$$q_B = q_{F\#} = q_{C\#} = 0$$

The remaining and obligate slightly augmented fifths impurities can finally be equally distributed over the fifths on Ab(G#), E*b*, B*b*, in order to define the pitches of B*b* and E*b* :

$$q_{Ab} = q_{Eb} = q_{Bb}$$

Observation: the ease of tuning could also have been achieved instead, by setting perfect fifths on the flat notes A*b*, E*b*, B*b*, and slightly augmented fifths on the notes B, F# and C#. The above proposed arrangement has the advantage that non of the natural notes of the diatonic C–major holds an augmented fifth.

[5.1.3] Obtained circular tempered meantone

The collection of solutions leads to the scale, table 6 :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	176.00	184.48	196.78	207.55	220.00	234.26	245.98	263.12	276.73	294.07	311.93	328.89
q_{Note}	– 1.77	0.00	– 2.21	1.23	– 2.21	1.23	0.00	– 2.21	0.00	– 2.21	1.23	– 2.76
p_{Note}	0.00	14.60	0.00	14.73	6.91	5.00	17.83	0.00	24.36	5.53	14.61	15.89

Table 6 : Circular Tempered Meantone ; Δ(cent) = 1.39

As was observed already, this circulating tempered meantone holds natural notes identical to those of the beating rate defined meantone, table 3.

[5.2] A WELL TEMPERED MEANTONE with a different division of the syntonic comma (auditory tuneable)

This paper, especially this paragraph, came about thanks to intense correspondence and close cooperation with A. Calvet, professional auditory tuner.

Tuning history shows a move between two extremes : perfect fifths for Pythagorean tuning, and just major thirds for meantone tuning. Therefor, when initiating the meantone tempering with the aim for well tempering, one could aim for a good balance between fifths and major thirds impurities, instead of setting a just major third built on diminished fifths.

A possible balance, easy for auditory tuning, is to set a major third C – E, with a beating rate impurity equal to that of the four fifths that built it. This leads to a system with 4 equations (q_{Note} and p_{Note} : see tables 1 and 2) holding 4 variables : 4 unknown notes (C, G, D, E) :

$$q_C = q_G = q_D = q_A = p_C$$

[5.2.1] The F and B notes

Such as for the classic meantone, one can go on setting major thirds. But instead of being just, those should now hold an equal beating rate impurity. The continuation of equal beating rates of major thirds on notes F and G means that we want :

$$q_C = q_G = q_D = q_A = p_C = p_F = p_G$$

This set of equations can be solved : it holds 6 equations with 6 variables : 6 unknown notes (F, C, G, D, E, B). Control of the solution shows that on top of the already desired impurity equalities, we obtain the same impurity also, for the fifth on E. This means we have, in effect :

$$q_C = q_G = q_D = q_A = p_C = p_F = p_G \quad = \quad q_E$$

This is a set of 7 equations with only 6 variables, but it can be solved indeed, because the seventh equation has been found redundant. The obtained note ratios are displayed below, and the obtained pitches and equal beating rates are on display in table 7 further on.

$$-q_{Note} = p_{Note} = \frac{A3}{113} = \frac{5F3}{451} = \frac{C4}{135} = \frac{G3}{101} = \frac{D4}{151} = \frac{E4}{169} = \frac{2B3}{253}$$

[5.2.2] The altered notes of C–major

The above obtained pitches for the F3 and B3 notes, embracing six diminished fifths, impose an average fifth ratio slightly exceeding 3/2, for the remaining six fifths.
This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{5 \times 253}{2 \times 451} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500396\,.\,.\,.$$

The above ratio is slightly above perfection, but even closer to perfection than in sections 4.1 and 5.1.2 above. The further definition of the altered notes can therefore be hold identical to the procedure applied in section 5.1.2.

[5.2.3] Obtained Well Tempered Meantone

The collection of solutions leads to the following scale, table 7 :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.61	184.71	196.64	207.80	220.00	234.02	246.28	262.83	277.07	293.98	311.90	329.03
q_{Note}	– 1.17	0.00	– 1.95	0.39	– 1.95	0.39	0.00	– 1.95	0.00	– 1.95	0.39	– 1.95
p_{Note}	1.95	12.51	1.95	12.32	8.27	5.84	16.17	1.95	19.54	7.79	13.62	17.28

Table 7 : Well Tempered Meantone ; Δ(cent) = 0.42

As can be observed, the Well Tempered Meantone holds 5 fifths and 3 major thirds with a defined equal beating rate, equal to – 1.95 . . . bps. This equality of beatings rates should not be confused with the equal (cent–) impurities of the 12TET[2].

Figure 1 holds a horizontal axes obtained by deployment of the circle of fifths, that is opened at the G# (= Ab) note. It displays a “conventional” representation of the fifths and major thirds impurities in cents, and a rather normal course can be perceived, very comparable to those of most famous well temperaments. All fifths are purer than the purest major thirds, and the impurity of slightly augmented fifths on A*b* (G#), E*b*, B*b*, turns out to be insignificant.

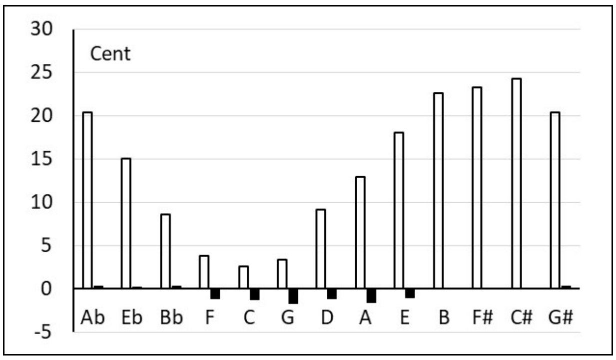

Figure 1 : Impurities of fifths (in black) and major thirds

Figure 2 holds a horizonal axes obtained by deployment of the circle of fifths, that is opened at the F3 note. It "unconventionally" illustrates the fifths and thirds impurities in beatings/sec, within a chromatic scale on F3, the scale usually used for auditory tuning. The notes on the figure are given in an "unconventional" inverted sequence of fifths. The displayed course is remarkable and regular for the natural notes of C–major, and illustrates very well the obtained impurity equality of some fifths and major thirds.

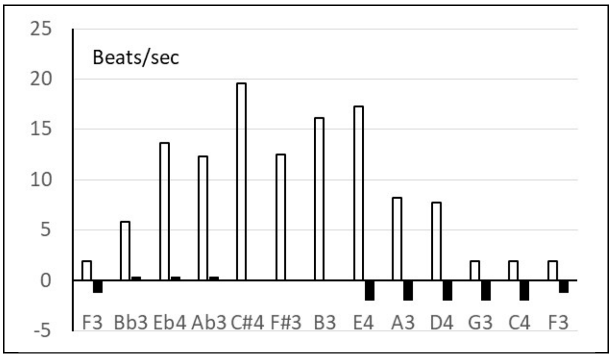

Figure 2 : Impurities of fifths (in black) and thirds

Figure 3 displays a clear and simple looking overview of the determined impurities distribution on the circle of fifths, whereby equality of markings signifies equality of interval beating rates :

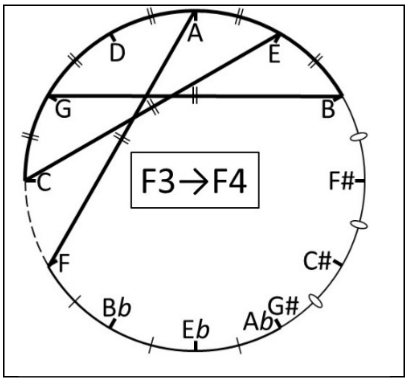

Figure 3 : Impurities distribution

[5.2.4] The well tempered meantone ; tuning practice

[5.2.4.1] The Well Tempered Meantone ; Chorton ("German organ diapason" ; A = 440)

The defined Well Tempered Meantone, table 7, paragraph 5.2.3, is still somewhat too complex for direct auditory application ; beating rates cannot auditory be measured up to two decimal digits precision. An alternative tuning instruction applied by Calvet holds fifths beating rates at – 2 , 0 , and 0.5 bps. Theoretical resulting pitches are displayed in table 8.

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.67	184.69	196.67	207.77	220.00	234.06	246.25	262.89	277.03	294.00	311.91	329.00
q_{Note}	– 1.23	0.00	– 2.00	0.50	– 2.00	0.50	0.00	– 2.00	0.00	– 2.00	0.50	– 2.00
p_{Note}	1.66	12.79	1.67	12.69	8.13	5.71	16.39	1.56	20.19	7.50	13.78	17.19

Table 8 : Well Tempered Meantone ; possible tuning practice for "Chorton" (A = 440)

The diatonic major thirds beatings become slightly slower, but remain mutually quasi equal.
It appears to be a very good and practical applicable temperament.

[5.2.4.2] The Well Tempered Meantone ;
Kammerton ("German chamber diapason" ; A = 415)

The defined Well Tempered Meantone, table 7, paragraph 5.2.3, can also be worked out for a diapason of A = 415 ; see table 9. This can be set based on the instruction to set fifths at beating rates "slightly below – 2 ", and at 0 , and 1/3 bps.

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	165.63	174.22	185.46	195.99	207.50	220.72	232.29	247.90	261.33	277.28	294.17	310.33
q_{Note}	– 1.10	0.00	– 1.84	0.37	– 1.84	0.37	0.00	– 1.84	0.00	– 1.84	0.37	– 1.84
p_{Note}	1.84	11.80	1.84	11.62	7.80	5.51	15.25	1.84	18.43	7.35	12.85	16.30

Table 9 : Well Tempered Meantone ; possible tuning practice for "Kammerton" (A = 415)

It appears to be a very good and practical applicable temperament.

[5.3] The WELL TEMPERED MEANTONE compared to Historic Temperaments

The defined Well Tempered Meantone, table 7, paragraph 5.2.3 appears to probably be the best possible auditory tuneable well temperament.

The Δ(cent) difference of this Well Tempered Meantone, with the "optimal" well temperament table 5, paragraph 4.1 amounts to only 0.42 cents. The Δ(cent) difference of the Vallotti temperament with the "optimal" temperament table 5 amounts to 0.87 cents, the next best is Barca (acc. Devie) with 1.18 cents, and all other historical temperaments have still higher differences. The Δ(cent) difference of the circular tempered meantone, table 6, paragraph 5.1.3, with the "optimal" well temperament table 5 amounts to 1.39 cents.

Observation :
Musical intuition might tend to accept higher beating rates for intervals on higher base notes.

This brings us back to the determination of notes based on cents, ratios or commas. A cents–"optimal" well temperament can be defined, with equal impurity expressed in cents for all diatonic C–major fifths and major thirds, in analogy with the optimal beat rate definition.

Based on this cents–"optimal" temperament we obtain that the Δ(cent) difference with this Well Tempered Meantone amounts to only 0.66 cents. The Δ(cent) difference of the Vallotti temperament with the cents–"optimal" temperament amounts to 0.35 cents, the next best is Barca (acc. Devie) with 0.84 cents, and all other historical temperaments have still higher differences.

The Δ(cent) difference of the circular tempered meantone, table 6, paragraph 5.1.3, with the cents–"optimal" well temperament table 5 amounts to 1.58 cents.

[6] "Das wohltemperirte Clavier" (The well tempered Keyboard)

Well temperaments become often related to Werckmeister, but those become even more often related also to "Das wohltemperirte Clavier" (1722 ; 1740 – 42), a masterpiece of J. S. Bach.

Discussions on Bach temperaments can be controversial, due to the fact that J. S. Bach has not left any written instructions on how to tune a keyboard. No historic certainty at all exists on how he tuned his clavichord, although it is mentioned and generally accepted he was very skilled at it, and extremely rapid.

It may not be forgotten though, that Bach's musical education was based on the meantone, and that he probably only had some initial experiences with some well tempering during his visit to

Buxtehude in Lubeck in 1705 (Kelletat, 19, p. 33, footnote 1). His well tempering might therefor have some links with the meantone, . . . question mark . . . ?

He was for sure very sensitive to musical affects : he enlightened the qualities of well temperaments by means of “Das wohltemperirte Clavier” (1722), but he also expressed horror in some parts of the “St. Matthew Passion” (1727), by intentional application of “forbidden” meantone keys (E, B, F#, C#, G#, Eb) on meantone tuned instruments (Kelletat 1982, p. 20).

Observation: the “St. Matthew Passion” (1727), where meantone can have a major impact on some musical affects, is POSTERIOR to “Das wohltemperirte Clavier” (1722) based on well tempering.

Bach’s clavichord tempering was discussed, sometimes indirectly, by Kirnberger (1771), in the letters of Kirnberger to Forkel (Kelletat, 1960, 1981, 1982), and by Forkel (1802), all arguing in favour of some kind of well temperament.

Bach’s clavichord tempering was also discussed by Marpurg (1776, par. 228, p. 213), arguing in favour of the application of equal temperament (12TET) by Bach. It must be noticed Marpurg invokes Kirnberger, the latter should as a Bach student have been referring to Bach regarding this topic. Kirnberger, however, contradicting its student Marpurg, strongly denies this privately in letters to Forkel (Kelletat 1981, p. 42, footnote 18). The Marpurg assumption concerning Bach’s temperament was copied by a countless number of authors in a countless number of publications, over two centuries, and even today.

A probably first doubt concerning the application of the 12TET by Bach was published by Bosanquet H. (1876, p. 28–30), and a breakthrough about those doubts probably came about through Kelletat (1960).

Kelletat assumes that the temperament that could have been applied by Bach could have been Kirnberger III, OR ANY OTHER SIMILAR ONE. Considering certain dates the latter opinion is probably preferable : "or any other similar one". Indeed, "Das wohltemperirte Clavier" dates from 1722, Kirnberger was a student of Bach from 1739 to 1741, and Bach died in 1750 ; the Kirnberger I temperament dates from 1761, Kirnberger II from 1771, and Kirnberger III from 1779.

Since Kelletat many assumptions are published about possible “Bach–temperaments”. To cite the best known ones only, we can (chronologically) think of Kelletat (1966), Kellner (1977), Billeter (1979), Sparschuh (1999), Zapf (2001), Jobin (2005), Lehman (2005), Lindley (2006), Amiot (2008), and many others for sure are missing in this summary list (Calvet and Lehman for instance, publish longer lists).

Figure 4 illustrates the fifths impurities course in bps, of the Well Tempered Meantone table 7, paragraph 5.2.3, in a same notes sequence as on figure 2. This is also the sequence in which they can be assumed on a scrolled figure from J. S. Bach, on a score of “Das wohltemperirte Clavier” : an “inverted” fifths sequence, within the F3 – F4 range, – this is the notes range also used for tuning –. The scrolled part of figure 4 is copied from Amiot (2008).

One can observe a striking parallelism of note marks for the Bach scrolls as well as for the Well Tempered Meantone fifths beatings course, on the graph above the scrolls. Possibly a closest fit, if compared with many other Bach–hypothesis.

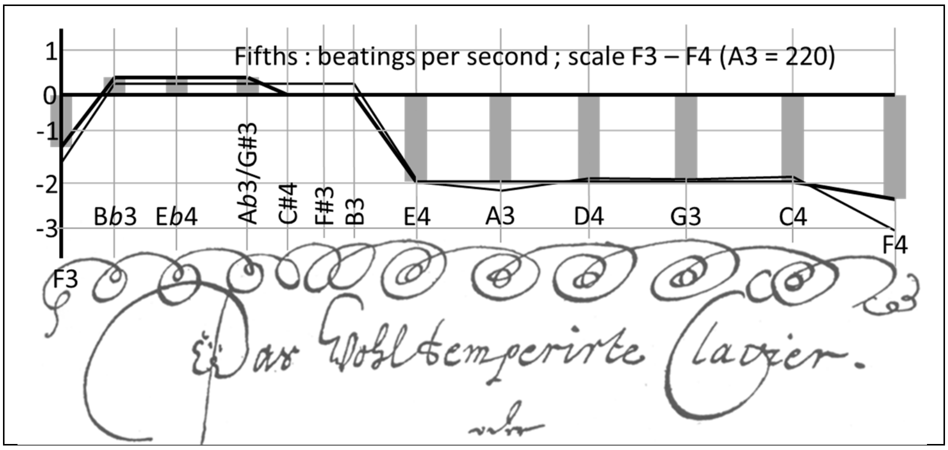

Figure 4 : "Well Tempered Meantone" fifths impurities, in bps. Fat line / grey bars (par. 5.2)
Slim line : "optimal" auditory tuning model (par. 4)

Remarkable also, is that this comparison is obtained following a Well Tempered Meantone construction based on auditory tuning and mathematical thoughts, where, on the other hand Bach was not interested in concerns about ratios or mathematics of temperaments, but the merely about (auditory) harmony and purity (Forkel, p. 39[3]) :

> << As purposeful and reliable Bach's style of teaching was in playing, so was it also in composition. He did not start with dry counterpoints that led to nothing, as was the case with other music teachers in his days ; still less did he hold up his students with calculations of the tonal relations which, in his opinion, did not belong to the composers, but the mere to theoreticians and instrument makers. >>

The exceptional dexterity and speed with which Bach could auditory tune a clavichord (Forkel, p. 17 ; Kelletat, 1981, p. 52 – 53) allows to assume that he very probably tuned the auditory way only.

If Bach should have tuned according the Well Tempered Meantone (very emphasized question mark; there is not yet historical evidence), than it could be said he solved in auditory way, a somewhat complex mathematic and auditory problem, of ***ease of auditory tuning, paired to well tempering, with an almost optimal diatonic C–major and with equal interval beating rates***.

[7] Conclusion

- The proposed Well Tempered Meantone, table 7, paragraph 5.2.3, offers a unique combination of ease of auditory tuning, paired to well tempering, with almost optimal diatonic C–major and with equal interval beating rates.
  This might be related to J. S. Bach, . . . question mark ?
- ***It is not the strict and exact mathematical equality*** of fifths and thirds impurities that prevails, . . . but that what prevails in this indeed, is the "auditory equality judgment" of the interpreting musicians and auditory tuners, their "brainear" ("cervoreille" ; calvet 2020) observing sounds as comprehensive coherent groups, leading to virtual equal beating impurity of some fifths and major thirds.

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Their open attitude allowed for quite intense exchange of ideas, that enabled further understanding and evolvement of insights in musical temperament and tuning matters, enabling the development of the ideas and concepts expressed in this paper.

Thanks to my daughter Hilde : she drew my attention to investigate on what musicians want (diatonic interval purity) and not on what might be someone's preferred musical temperament.
DEDICATION

This paper is dedicated to all classic musicians and auditory tuners.
Their sensitive musical ears offer to our world all the best of the most universal and most beautiful of all languages : ***MUSIC***.

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https://home.deds.nl/~broekaert/Index.html

WORKS CITED

Note : the list below includes all references of a more detailed and extensive version of this paper, from the same author.

Amiot E. 2008 : "Discrete Fourier Transform and Bach's Good Temperament"
https://mtosmt.org/issues/mto.09.15.2/mto.09.15.2.amiot.html

Barbour J. M. 1951 : "Tuning and Temperament: A Historical Survey"

Bosanquet H. 1876 : "An elementary treatise on musical intervals and temperament"
https://en.xen.wiki/images/a/a7/Bosanquet_–_An_elementary_treatise_on_musical_intervals.pdf

Calvet A. 2020: "Le Clavier Bien Obtempéré"
http://www.andrecalvet.com/v3/index.php

De Bie J. 2001 : "Stemtoon en stemmingsstelsels", private edition, most data originate from
Barbour J. 1951: "Tuning and Temperament: A Historical Survey"

de Zwolle H. A. (~1400 – 1466) : « Liber proportionum musicalium » (~1450).

Devie D. 1990 : "Le Tempérament Musical : philosophie, histoire, théorie et pratique"

Douwes C. 1699 : "Grondig onderzoek van de Toonen der Musyk",

Fritz B. 1756 : "Anweisung, wie man Claviere ... in allen zwölf Tönen gleich rein stimmen könne …"
https://gdz.sub.uni– goettingen.de/id/PPN630630391

Forkel J. 1802: Ueber Johann Sebastian Bachs Leben, Kunst und Kunstwerke.
https://www.digitale-sammlungen.de/en/view/bsb10528130?page=,1

Jedrzejewski F. 2002 : "*Mathématiques des systèmes acoustiques, Tempéraments et modèles contemporains.*" L'Harmattan, Paris, 2002.

Jobin E. 2005 : "BACH et le Clavier bien Tempéré" ;
https://www.clavecin– en– france.org/spip.php?article52.

Kelletat H. 1957 : "Ein Beitrag zur Orgelbeweging. Vom Klangerlebnis in nichtgleichswebenden Temperaturen"

Kelletat H. 1960 : "Zur musikalischen Temperatur"

Kelletat H. 1981 : "Zur musikalischen Temperatur ; Band I. Johann Sebastian Bach und seine Zeit"

Kelletat H. 1982 : "Zur musikalischen Temperatur ; Band II. Wiener Klassik"

Kelletat H. 1993 : "Zur musikalischen Temperatur ; Band III. Franz Schubert"

Kellner H. 1977 : "Eine Rekonstruktion der wohltemperierten Stimmung von Johann Sebastian Bach"

Kroesbergen W. 2013 : "18th Century Quotes on J.S. Bach's Temperament" see :
https://www.academia.edu/5210832/18th_Century_Quotations_Relating_to_J.S._Bach_s_Temperament

Lehman B. 2005 : "Bach's extraordinary temperament: our Rosetta Stone – 1 ; – 2" ; Early Music

Marpurg F. 1776: "Versuch über die musikalische Temperatur"
http://www.deutschestextarchiv.de/book/view/marpurg_versuch_1776?p=5

Napier J. 1614 : "Mirifici logarithmorum canonis descriptio"

Norback J. 2002 : "A Passable and Good Temperament ; A New Methodology for Studying Tuning and Temperament in Organ Music", https://core.ac.uk/download/pdf/16320601.pdf

Pitiscus B. 1603 : "Thesaurus mathematicus"

Railsback O. 1938 : "Scale temperament as applied to piano tuning" J. Acoust. Soc. Am. 9(3), p. 274 (1938)

Rameau J.– P. 1726 : Nouveau système de musique théorique

Rheticus G. 1542 : "De lateribus et angulis triangulorum (with Copernicus ; 1542)"

Salinas F. 1577 : "De musica libra septem"
https://reader.digitale– sammlungen.de//resolve/display/bsb10138098.html

Sparschuh A. 1999 : "Stimm– Arithmetic des wohltemperierten Klaviers von J. S. Bach (TU Darmstadt)

Stevin S. ca. 1586 : "De Thiende" (The Tenth)

Stevin S. ca. 1605 : "Vande Spiegheling der Singconst" (Considerations on the art of singing)
https://adcs.home.xs3all.nl/stevin/singconst/singconst.html

Werckmeister A. 1681 : "Orgelprobe"

Werckmeister A. 1686 : "Musicae Hodegus Curiosus"
http://digitale.bibliothek.uni– halle.de/vd17/content/titleinfo/5173512

Werckmeister A. 1689 : "Orgelprobe"
https://reader.digitale– sammlungen.de/de/fs1/object/display/bsb10527831_00007.html

Werckmeister A. 1691 : "Musicalische Temperatur"
https://www.deutsche– digitale– bibliothek.de/item/VPHYMD3QYBZNQL2UAK3Q2O35GBSVAIEL

Zapf 2001: "Handling Down the Tradition: The Survival of Bach's Fingering Technique in an Obscure Nineteenth–entury Clavier Tutor" (De Clavicordio, sept. 2001, pr. 39–44).

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[1] Original German definition :
"Wohltemperierung heißt mathematisch-akustische und praktisch-musikalischen Einrichtung von Tonmaterial innerhalb der zwölfstufigen Oktavskala zum einwandfreien Gebrauch in allen Tonarten auf der Grundlage des natürlich-harmonischen Systems mit Bestreben möglichster Reinerhaltung der diatonische Intervalle.
Sie tritt auf als proportionsgebundene, sparsam temperierende Lockerung und Dehnung des mitteltönigen Systems, als ungleichschwebende Semitonik und als gleichschwebende Temperatur."

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[2] Misunderstanding might be possible in the German language : the here obtained equal beating corresponds to “Gleichschwebend”, whereby on the other hand the 12TET is named by “Gleichschwebende Temperatur”. The same confusion is possible too in the Dutch language.

[3] Original German text :
So zweckmäßig und sicher Bachs Lehrart im Spielen war, so war sie es auch in der Composition. Den Anfang machte er nicht mit trockenen, zu nichts führenden Contrapuncten, wie es zu seiner Zeit von andern Musiklehrern geschah; noch weniger hielt er seine Schüler mit Berechnungen der Tonverhältnisse auf, die nach seiner Meynung nicht für den Componisten, sondern für den bloßen Theoretiker und Instrumentenmacher gehörten.