Proposal for a Comprehensive Evaluation of Bach– and Well–Temperaments

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Abstract

A musical temperament impurity measure formula is worked out. It appears to lead to meaningful results regarding temperament evaluations. Further on, the ranking results fit very well with breakthrough historical and musicological research of professor H. Kelletat, whom demonstrates which essential characteristics should be proper to so-called candidate "Bach temperaments".

Keywords

Well temperament ; circulating temperament ; meantone ; evaluation ; auditory tuning ; keyboard ; interval ; impurity ; diatonic ; Bach

[1] Brief Bach–Temperament History

A brief Bach temperament history goes necessarily at par with a well temperament history. Where meaningful, well temperament facts will briefly be mentioned too.

1681 A. Werckmeister publishes "*Orgelprobe*", and mentions "wohl" and "temperiren" on the frontpage, what later on will lead to the German "wohltemperiert" term (= well tempered). It is assumed that well-temperaments existed before this publication. Werckmeister uses the same terms also later on (1686, p. 118 (=108) and 120 ; 1689, titlepage and p. 61 ; 1698. p. 7).

1722 / 1740–1742 J. S. Bach publishes "*Das wohltemperirte Clavier*" (WTC).

1771 J. Kirnberger, a Bach pupil, publishes "*Die Kunst des reinen Satzes in der Musik*", he claims later on, that it reflects Bach's teaching (1782 p. 3 and 5; Forkel, par. VII p. 41). He explains his Kirnberger III temperament in 1779, in a letter to Forkel (Kelletat, 1981, p. 37 footnote 63 ; 1982, p. 140)

1776 F. Marpurg publishes "*Versuch über die musikalische Temperatur*". He pretends (p. 212–213), referring to Kirnberger, his teacher, that Bach should have used and taught the use of the "Gleichschwebende Temperatur" [equal temperament = 12TET (= 12 equal fifths)]. Kirnberger very strongly denies, what is clear from his letters to Forkel (Kelletat 1981, footnote 20, p. 42), but he prefers further to keep his opinion private (Kelletat 1981, footnote 19, p. 42).

The Marpurg opinion concerning Bach and his application of the 12TET becomes accepted and is copied in countless publications over a period of two centuries, from Forkel to (Bosanquet)–Kelletat.

The Baroque period was a very fertile period in history, all over Europe, regarding the creation of well temperaments ; think of Vallotti, Mercadier, Neidhardt, Lambert, Sorge, etc. …

1802 J. Forkel publishes "*Über Johan Sebastian Bach's Leben, Kunst und Kunstwerke*". Forkel was friend with Bach's suns Carl Philipp Emanuel and Wilhelm Friedemann, and also with Kirnberger. He testifies (par. VII p. 41) that Kirnberger's publication reflects indeed Bach's teaching, and this was not contradicted by Bach's suns..

1876 R. Bosanquet publishes "*An elementary treatise on Musical Intervals and Temperament*". He has strong doubts that Bach could have used the 12TET (p. 29–30), but at his time his message is not really "captured" as being relevant.

1960 H. Kelletat publishes "*Zur musikalischen Temperatur*". *Probably* t*he very first publication after Bosanquet, expressing strong doubts that Bach could have used the 12TET.* His publications (1960, 1981, 1982, 1994) rely on numerous citations (more than 1000) providing historical and musicological evidence on that point. *The Kirnberger III temperament, or some other comparable one, must have been the temperament applied by Bach* (Kelletat, 1982, p.141).
From the above hypothesis "*Kirnberger III, or some other comparable one*", the comparable one probably has to be preferred : Bach's WTC is published 1722 / 1740–42, and the Kirnberger (I, II, III) temperaments are published in 1766 , 1771 and 1779, quite far later than the WTC.
At this point, because of the factual dates, one could assume, for example, that Kirnberger used another temperament before 1766 , probably the one that he might have been using with Bach, –one that should have been simple to tune (Forkel, par. III, p. 17)–, but that later on he might have been seeking for *a different one of same quality, but easier still to tune than Bach's one* ?
It must be admitted, there is no written source that could confirm this hypothesis.

1966 H. Kelletat publishes his "Bach temperament" version (Kelletat, 1982, table 9, p. 155 ; table 12, p. 158).

1977 H. Kellner ; 1979 B. Billeter ; 1979 J. Barnes : Bach temperament proposals, see works cited

1981 H. Kelletat publishes "*Zur musikalischen Temperatur I. Johann Sebastian Bach und seine Zeit*". Same content as the 1960 publication.
1982 H. Kelletat publishes "*Zur musikalischen Temperatur II. Wiener Klassik*". A number of temperaments receive the attribute of "Bach temperament": Kirnberger III, Kelletat, Kellner and Billeter, whereby it is claimed it is impossible to distinct these from one another by the ear (observation 3, p. 142)

From now on many "Bach Temperaments" are proposed. The following listing is for sure not complete.

1994 M. Lindley: hypothesis based on strong historical and musicological considerations ; see works cited
1998 A. Sparschuh: "*Stimm–Arithmetic des wohltemperierten Klaviers von J. S. Bach*". This is probably the first approach to reconstruct a Bach–Temperament based on a curled figure, drawn by J. S. Bach, on the top of a score of "*Das wohltemperirte Clavier*". Alternatives are proposed later on.
2000 M. Jira: see works cited
2001 M. Zapf: see works cited. It concerns some variation on Sparschuh's proposal.
2004 J. C. Francis: see works cited. Inspired by Sparschuh.
2005 B. Lehman: see works cited. Inspired by the Bach scrolls, differing from Sparschuh's proposal.
2005 P. Allain-Dupré: see works cited
2005 E. Jobin: A new interpretation of the Bach scrolls, based on the meantone and cent calculations.
2006 J. O'Donnell ; 2006 M. Spanyi: see works cited.
2007 G. Interbartolo, P. Venturino: see works cited
2008 C. Di Veroli ; 2008 E. Amiot ; 2020 J. Broekaert : see works cited

[2] Evaluation of Well Temperaments

It is not easy to define a comprehensive algorithm for well temperament evaluation (Hall D. ; p. 275-277). Most research and evaluations on temperament characteristics are based on the evaluation of interval properties or comma divisions, measured in proportions or cents. Some “subjective” musical discussions on peculiar characteristics are very often part of an evaluation.
Measurement of interval proportions is common musicological practice since Pythagoras at least. It has been used intensively during the Baroque period, based on monochord measurements, and later on calculations are facilitated by the introduction of the cent, a logarithmic measure of proportions. And still, to musicians and auditory tuners those numbers are rather abstract, and it are mainly the perceptible beatings that import to them. *Those beatings are relatable to good harmonic properties such as the consonance and dissonance of intervals and musical chords*. Some further investigation on interval beating rate properties might therefor be of interest, regarding auditory musical keyboard tuning and comprehensive temperament characteristics.

[2.1] Measurement of Interval Purity

The auditory tuning of a musical keyboard is normally based on the evaluation of fifths and major thirds beating rates, within the chromatic scale from F3 to F4 (Calvet A.). To enable auditory tuning, one has to dispose of a table, listing the beating rates of fifths. The beating rates of some thirds might import too, mainly to control the precision of the tuning quality.
The beating rates of the fifths and major thirds on the notes within the F3–F4 chromatic scale can be calculated by means of the formulas in the tables below. The q_{Note} and p_{Note} symbols stand for the beating rates of fifths and major thirds. The formulas for q_C and p_C were applied by A. Kellner (1977), and he applied a similar formula for the minor third on E also.

$q_F = 2C4 - 3F3$	$q_C = 4G3 - 3C4$	$q_G = 2D4 - 3G3$	$q_D = 4A3 - 3D4$
$q_A = 2E4 - 3A3$	$q_E = 4B3 - 3E4$	$q_B = 4F\#3 - 3B3$	$q_{F\#} = 2C\#4 - 3F\#3$
$q_{C\#} = 4G\#3 - 3C\#4$	$q_{G\#} = 2Eb4 - 3G\#3$	$q_{Eb} = 4Bb3 - 3Eb4$	$q_{Bb} = 4F3 - 3Bb3$

Table 1 : calculation of fifths beating rates within the F3 – F4 scale

$p_F = 4A3 - 5F3$	$p_C = 4E4 - 5C4$	$p_G = 4B3 - 5G3$	$p_D = 8F\#3 - 5D4$
$p_A = 4C\#4 - 5A3$	$p_E = 8G\#3 - 5E4$	$p_B = 4Eb4 - 5B3$	$p_{F\#} = 4Bb3 - 5F\#3$
$p_{C\#} = 8F3 - 5C\#4$	$p_{G\#} = 2C4 - 5G\#3$	$p_{Eb} = 8G3 - 5Eb4$	$p_{Bb} = 4D4 - 5Bb3$

Table 2 : calculation of major thirds beating rates within the F3 – F4 scale

[2.2] Well Temperaments

A possible musical definition of Well Temperaments is given by Kelletat (1960 ; 1981, p. 9 ; original German text : see endnote [A]), based on Werckmeister (1681, 1689):

> *<< Well temperament means a mathematical–acoustic and musical–practical organization of the tone system within the twelve steps of an octave, so that impeccable performance in all tonalities is enabled, based on the extended just intonation (natural–harmonic tone system), while striving to keep the diatonic intervals as pure as possible.*
> *This temperament acts, while tied to given pitch ratios, as a thriftily tempered*

smoothing and extension of the meantone, as unequally beating half tones and as equal (equally beating) temperament. >>

[2.3] A Well Temperament Optimum

A large collection of well temperaments is available. One might wonder which ones could be among the bests, based on the above definition par. [2.2]. It might therefor be useful to define at first some kind of limit corresponding to a mathematical optimum temperament, for comparison purposes.
A major requirement in line with the definition paragraph [2.2] consists in keeping the diatonic intervals of the C–major natural–harmonic tone system as pure as possible. The auditory tuner will therefor mainly evaluate the C–major diatonic fifths (on C, D, E, F, G, A) and major thirds (on F, C, G). An intuitive approach to this requirement might be to calculate the lowest possible impurity of those intervals. Mathematically this corresponds to the minimization of the following sum:

$$p_C^2 + p_D^2 + p_E^2 + p_F^2 + p_G^2 + p_A^2 \quad + \quad q_C^2 + q_F^2 + q_G^2$$

Following an advice from E. Amiot (professor mathematics and musician), the minimum of this sum can swiftly be determined by calculation of the partial derivatives to the variables (the unknown notes), set to equal zero, followed by the solution of the set of obtained equations (see par. [5] ; appendix 1).
The above procedure leads to a temperament holding fifths beating rates from –0.89 to –3.16 beatings per second (bps.). This is not convenient to an auditory tuner. He has indeed to know and set nine differing beating rates, or, if those are not known, he has to measure every single beating rate, calculate their sum, and check if this obtained sum can be minimized further on. This is very hard, iterative and difficult precision labor, and for sure not the common practice.

But it is easy, for any musician or tuner, to evaluate interval beating rate (= impurity) equalities. Obtaining a best possible equality corresponds mathematically to the minimization of the sum of all the differences between any single impurity and the mean value of all impurities. There are 9 intervals to be defined, for only 6 variables (the C–major diatonic notes but A). It is normally not possible to solve such a set of conditions. We can only strive for a minimum difference (see par. [6], appendix 2).

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.67	184.73	196.60	207.98	220.00	234.14	246.22	262.75	277.22	293.96	312.10	328.93
q_{Note}	– 1.52	0.26	– 1.89	0.26	– 2.15	0.26	0.26	– 1.83	0.26	– 1.87	0.26	– 1.89
p_{Note}	1.65	12.91	1.89	11.07	8.90	5.13	17.30	1.97	19.23	8.06	12.30	19.23
cents	12.28	–0.15	9.51	4.88	0.00	9.79	–2.62	10.31	2.48	2.18	7.44	–3.01

Table 4 : scale with optimal beating rate equality of the of diatonic fifths and major thirds within C major

The obtained average beating rate, for C–major diatonic fifths and major thirds between F3 and F4 , is – 1.85... bps., with minor deviations only from this value (≤ 0,33) ; see table 4.
This scale might be auditory tunable, if one reckons with slight diminishment or augmentation on F3 and A3 , but it must be admitted it still won't be easy.
Fortunately, a very close and easily auditory tunable alternative can be thought of : see the "Well Tempered Meantone", at par. [2.4] below.

[2.4] A Well Tempered Meantone

Inspired by Jobin, and the above obtained optimal and auditory tunable scale, a very practical and pure temperament can be defined, holding five diatonic fifths and the three diatonic thirds with *exact mathematical equality of their beating rates. The beating rate equality of fifths and major thirds is a nice musical compromise, between the perfect fifths of the Pythagorean tuning and the just major thirds of the quarter comma meantone tuning.* To do so we have to set:

$$q_C = q_G = q_D = q_A = -p_C = -p_F = -p_G \quad = \quad q_E$$

The above equations are a set of seven equations holding only six variables ; most often such sets cannot be solved. This set can be solved though, as it has been found the set is redundant: one of the requirements, either that of q_E or p_G , can be discarded.
The solution is very simple, and could even, the way it is formulated below, also look acceptable to Baroque musicologists, familiar with proportions:

$$-q_{Note} = p_{Note} = \frac{A3}{113} = \frac{5F3}{451} = \frac{C4}{135} = \frac{G3}{101} = \frac{D4}{151} = \frac{E4}{169} = \frac{2B3}{253}$$

Further on three (easily tunable) perfect fifths (on B, F#, C#), and three equally and very slightly augmented ones (on A*b*, E*b*, B*b*) can be set, to complete the scale. This remains in line with Jobin, whom, coming from the meantone, assumed the same fifths characteristics, but with the augmented ones being a little more augmented.

$$q_B = q_{F\#} = q_{C\#} = 0 \quad \text{and} \quad q_{Ab} = q_{Eb} = q_{Bb}$$

The collection of solutions leads to the following scale, table 5 :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.61	184.71	196.64	207.80	220.00	234.02	246.28	262.83	277.07	293.98	311.90	329.03
q_{Note}	– 1.17	0.00	– 1.95	0.39	– 1.95	0.39	0.00	– 1.95	0.00	– 1.95	0.39	– 1.95
p_{Note}	1.95	12.51	1.95	12.32	8.27	5.84	16.17	1.95	19.54	7.79	13.62	17.28
cents	9.85	–2.67	5.64	1.24	0.00	6.94	–4.62	7.96	–0.71	1.87	4.27	–3.16

Table 5 : Well Tempered Meantone

Auditory tuning of this temperament is easy indeed: it holds eight equal beating rates ; see the shaded cells of table 5 and the intervals on fig. 1 marked by " **=** ". The remaining fifths except the one on F are perfect (**0** mark), or almost perfect (**–** mark) ; the fifth on F (dotted line) is "schismatic" ("residual").
Kelletat emphasizes the importance of natural and chromatic semitones sizes, all over his four books (1960, 1981. 1982, 1994). Some similarity with meantone semi tones is required, and to his opinion, the Kirnberger III semi tones are satisfactory for Bach's music (1960, 1981), Viennese classical music (1982) and Schubert's "Liedern" (1994).
Semitone characteristics are graphically on display in fig. 2.

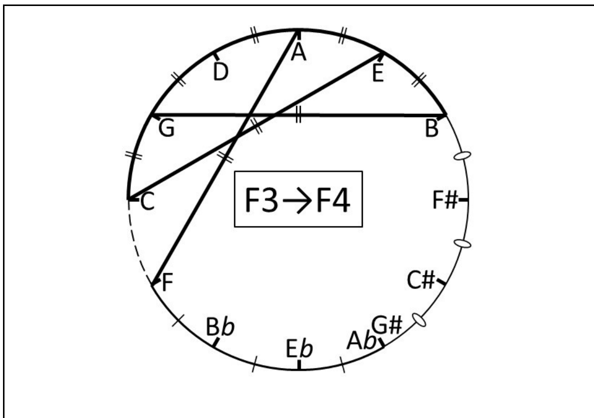

Fig 1: Well Tempered Meantone ; Circle of fifths

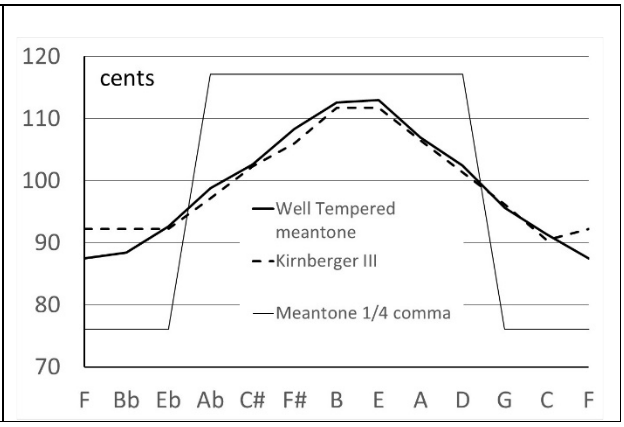

Fig. 2: Well Tempered Meantone ; Semi tone Characteristics

[3] Evaluation of Temperaments

[3.1] Numerical Evaluation

A probably novel evaluation method for musical temperaments is proposed, based on the RMS (the Root of the Mean of the Squares) measurement of C-major diatonic impurity differences (see par. [6], appendix 2). This measures the spread of the impurity deviations from the mean impurity, expressed in bps. To normalize the result we multiply the RMS by 440/A4 whereby A4 is the pitch of the used diapason. This leads to the following diatonic impurity definition (see the definition of the sum at par. [6], appendix 2):

$$\text{Diatonic Impurity} = \frac{440}{A4} \times \sqrt{\frac{\sum \Delta^2_{Fi\ and\ Th.Diatonic\ note}}{9}}$$

Many temperaments show very good diatonic purity, but many of those do not satisfy some "auxiliary" Well Temperament purity conditions. Therefor following conditions were selected to discard non well tempered temperaments:

1. Augmentation of fifths: augmented fifths are normally not accepted. To limit allowable augmentation, we take the obtained total fifths augmentations of the calculated optimum (table 4 ; = 3.7 cents) into account ; this corresponds to the condition whereby this full augmentation is set on only one fifth. This leads to a rounded value equal to the sum of a perfect fifth plus said total augmentation : this is 702.0 + 3.7 ≈ 706.0 cents.
2. Diminution of fifths: the Kirnberger II temperament was rejected, because of excessive diminution of fifths (Kelletat 1981, p. 47). The smallest Kirnberger II fifth measures 691.2 cents. As limit on the diminution of fifths, a minimum value of 692.0 cents was set.

Half of 140 historical temperaments had to be discarded because of the above conditions. The retained temperaments, are ranked by their obtained impurity figure, in table 6:

Optimal beating equality	0.170	Stanhope bps	1.562	Neidhardt-2	2.274
Well Tempered meantone	0.245	Mercadier bps	1.567	Asselin	2.321
Sievers	0.952	Neidhardt-4	1.568	Neidhardt 2 bps	2.356
Vallotti - Tartini	1.021	Stanhope	1.589	Sparschuh 1999 (b)	2.374
Vallotti bps	1.094	Neidhardt 1 bps	1.589	Neidhardt 3 bps	2.379
Venturino 1/4 1/19 (b)	1.156	Barnes	1.641	Sorge 1758 bps	2.379
Venturino 1/4 1/12 (b)	1.156	Jencka (b)	1.641	Sorge 1744 bps	2.400
Kirnberger III bps	1.171	Maunder b (b)	1.641	Bendeler III bps	2.410
d'Alembert / Rousseau	1.193	Lehman_1_6_Pyth (b)	1.641	Neidhardt-3	2.426
Legros (2 R.T.)	1.217	Werckmeister III bps	1.652	Sorge1728	2.426
Jobin	1.217	Neidhardt-1	1.663	Francis 2005 E (b)	2.430
Kirnberger III	1.235	Lehman bps	1.694	Sorge1744	2.441
Kirnberger III ungleich	1.246	Werckmeister III	1.737	Bendeler-III	2.542
de Bethisy	1.265	Lambert 1774	1.755	Bendeler-I	2.542
Kelletat	1.298	Weingarten / Gabler	1.834	Bendeler-II	2.592
Kellner bps	1.303	Barca (Asselin)	1.838	Sparschuh-Zapf (b)	2.835
Billeter	1.349	Di Veroli WTC opt (b)	1.887	Werckmeister II	3.054
Kellner	1.353	Mobbs/Mackenzie (b)	1.902	Romieu -1/9 sc	3.059
Young 1800	1.426	Lindley 1994 Neidhardt	1.921	Meantone -1/9 c	3.059
Barca (Devie)	1.445	Jira offen 1 (b)	1.967	Romieu -1/10 sc	3.386

Lindley 1994 Michaelstein (b)	1.499	Jira geschlossen 2 (b)	1.967	Barthold Fritz	3.510
Mercadier	1.502	Young / Van Biezen	2.074	***12-TET***	3.653
Neidhardt 4 bps	1.547	Francis 2005 1/14pc (b)	2.239		
Venturino 1/4 1/12 (b)	1.559	Maunder c (b)	2.239		

Table 6 : Well temperaments ranked against diatonic purity

Some of the temperaments of table 6 are marked by "(b)" , because of having been announced as "Bach temperament" ; some are marked by "bps" : those are recalculated temperaments, based on equal beating rate division of the comma, instead of the normal proportional division.

It is very remarkable that on the top eighteen of the list, nine ***bold-italic*** marked temperaments can be found that received the "Bach" predicate by Kelletat (1982, p. 141-142) ; Kelletat pretends that differences between those temperaments should not be perceivable by the ear (observation 3, p. 142).
Kelletat (1960, 1966, 1981, 1982, 1994) caused a historical breakthrough that lead to quite some numerous additional "Bach Temperament" proposals (see the brief history, par. [1]), *following his very strong historic and musicological demonstration that the Kirnberger III temperament or any other comparable one must have been used by J. S. Bach, and not the 12 TET, as was assumed for almost two centuries.*

Also remarkable, is the position of Lindley 1994: this temperament is a mathematical theoretical model, auditory tested on a conference, and worked out based on musicological and historical grounds, and on proportional comma divisions (instead of beating rate distributions).

Some temperaments marked by (b), supposed thus to be "Bach Temperament", do not have a favorable ranking. It can be observed that (b) marked temperaments are quite dispersed on the list, what could mean that there could have been insufficient concern on C–major diatonic purity in the determination of some of those temperaments.

The ***12 TET***, very remarkably, is ranked at the very end of the list.

The top of the ranking of the discarded temperaments might also be interesting :

Vogel (Norden)	0.425	Silbermann	1.021	Schlick ~ Husmann/Lange/Ratte	1.156
Meantone -1/5 s. c.	0.567	Meantone bps	1.077	Meantone 1/4 comma	1.156
Louet	0.567	Geib / Neu-Bamberg	1.107	Rameau in F	1.156
Vogel (Stade)	0.608	Lambert / Chaumont	1.156	Rameau sec. TLA	1.156
Gabler	0.680	Corrette	1.156	Marpurg	1.156
Vogel bps	0.724	Schlick / Dupont	1.156	Rameau	1.156

Table 7 : Non circulating temperaments sorted against diatonic purity (top of the list only)

The very high purity of the first six temperaments is remarkable.
The famous Silbermann and meantone temperaments have a favorable position.
Most remarkable is the position of the Meantone –1/5 s. c. This temperament was discussed by L. Rossi (1666) and J. Sauveur (1701). Its diatonic properties of the C–major tonality are very close to those of the Well Tempered Meantone and the "optimal beating equality" temperament. The 1/5 meantone has a 1/5 s.c. diminution of perfect fifths, at par with a 1/5 s.c. augmentation of just major thirds, a nice balance therefore between almost perfect fifths and almost-pure thirds.. This can be perfectly compared against the equal beating rates of fifths and major thirds of the Well Tempered Meantone. From this point of view, it is not impossible that at Baroque time some kind of well

temperament might have been derived from the Meantone –1/5 s.c., but no written documents on this hypothesis have been found or discussed yet.

[3.2] Graphic Evaluation

Bach temperaments, based on historical and musicological considerations, according Kelletat or Lindley, have quite typical characteristic courses of fifths, especially also of thirds. Those are on display in figure 3, that has a conventional ordinate, calibrated in cents (not in bps), and an abscise with conventional fifths sequence, for a chromatic scale from C to C .
The thirds course is the flattest for Lindley.
The thirds for B, F# and G# deviate most from purity, for the Well Tempered Meantone.
Kirnberger III is very acceptable in comparison to the three others.

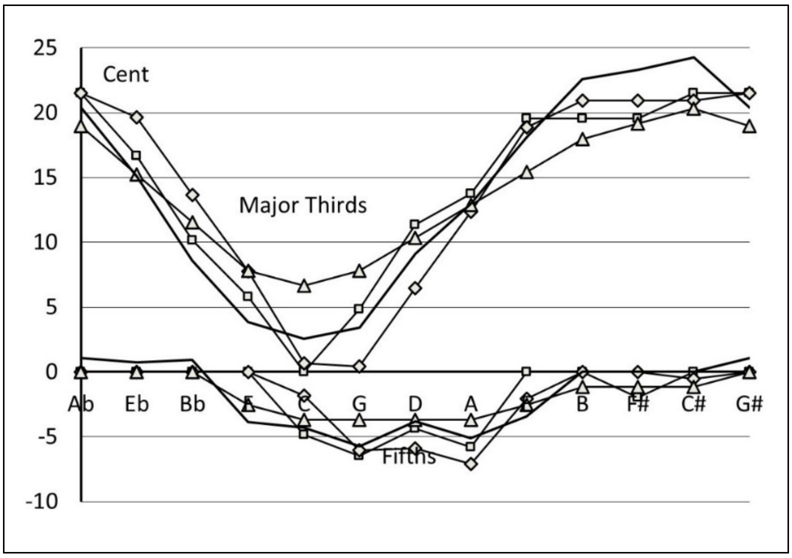

Fig 3 : Interval Impurities : no marks = Well Tempered Meantone ; squares = Kirnberger III ; diamonds = Kelletat ; triangles = Lindley

Figure 4 displays the courses of the 13 first temperaments of table 6, with an ordinate axes calibrated in bps (not in cents), and with a shifted and inverted abscise axes, that fits better to auditory tuning praxis, because it is for a chromatic scale from F3 to F4 , and it therefor has the F note at the extremes. It can be observed that the “bandwidth” within which the courses flow is quite restricted. Because of the bps calibration of the ordinate, it is easy to observe beating rate equality of five fifths (C, G, D, A, E) and three major thirds (F, C, G) of the Well Tempered Meantone.

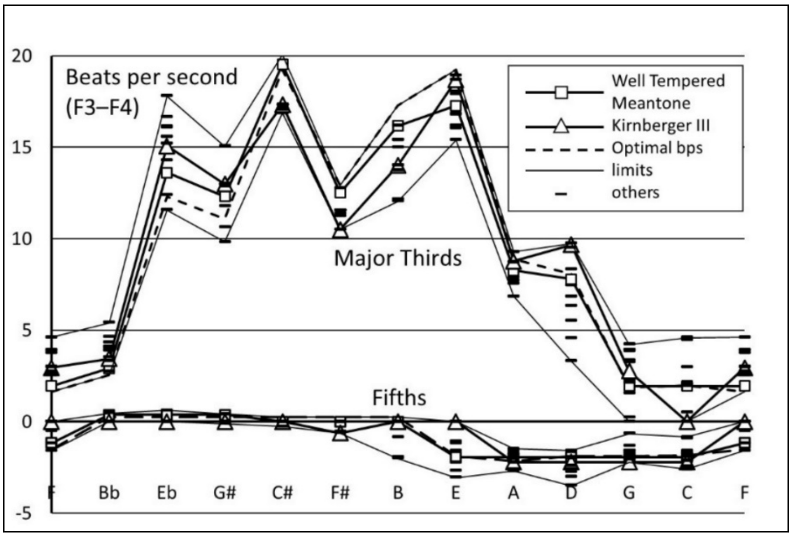

Figure 4: Well Temperaments (Bach selection)

Fig. 5 displays the similarity of the C-major diatonic fifths and thirds of the Well Tempered Meantone with those of the 1/5 s.c. meantone. It becomes clear that the completion of the tuning for the altered notes of the Well Tempered Meantone relies on purity of fifths, instead of the purity of major thirds such as applied for the 1/5 s.c. meantone.

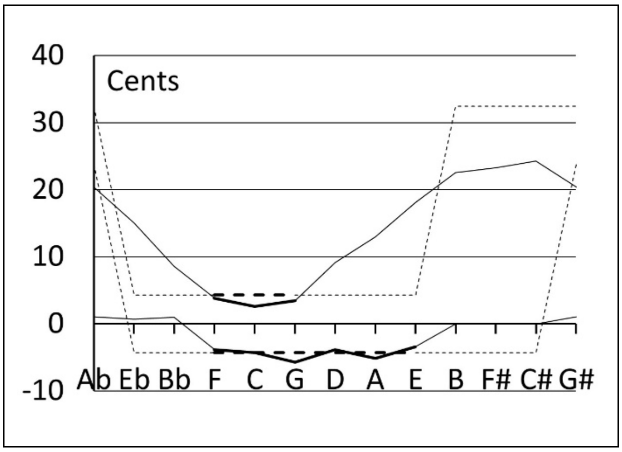

Fig 5. : Comparison of C–major diatonic impurities of the Well Tempered Meantone and the 1/5 s.c. meantone.

Since Sparschuh a number of hypothesis were formulated, based on pairing fifths characteristics with graphic characteristics of curls on Bach's WTC score, see figure 6 below.
It is not sufficient to rely on this type of analysis only, to demonstrate some kind of evidence that the compared temperament can be related to Bach. In table 6 for example, there is quite some distance between the hypothesis of Jobin (ranked on position 7/52 with an impurity of 1.217) and Lehman (on position 24/52 with an impurity of 1.641), while both are claimed to be a "Bach-temperament", based on the perception of the curls. The Sparschuh and Zapf hypothesis are still more different from Jobin, have undergone a number of revisions, and it remains difficult to match those with a plausible historical diapason.

Nevertheless, it is worth to make this comparison for the Well Tempered Meantone too, see fig. 6. It can be observed it matches the curls very well, just like for Jobin (not drawn on the figure} according to which is was inspired, and the optimal scale (thin line) fits quite well also.

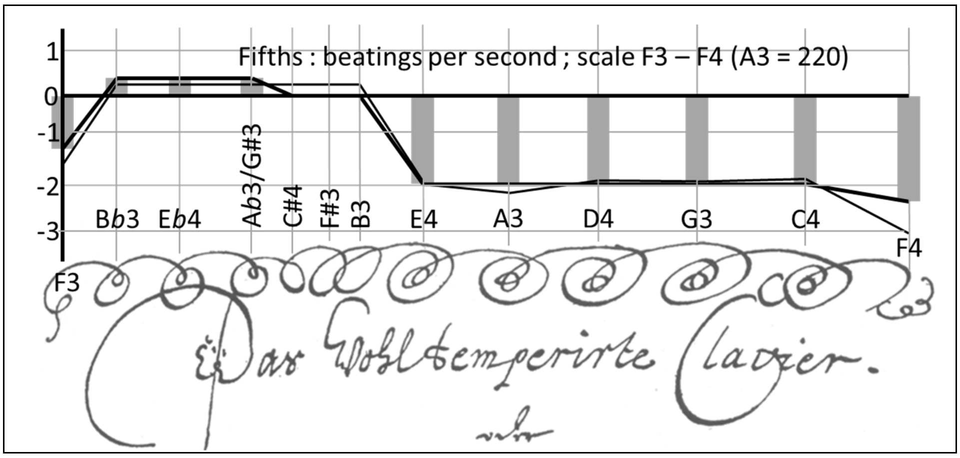

Figure 6 : "Well Tempered Meantone" fifths impurities, in bps. Fat line / grey bars (table 4)
Slim line : "optimal" auditory tuning model (table 5)

[3.2]` Summary of the numerical and graphical evaluation

The summary of the numerical and the graphical evaluation, supports an hypothesis as if the temperament used by Bach could have been derived indeed from the meantone, as suggested by the definition of Kelletat par. [2.2], probably the 1/5 comma meantone that has excellent C–major diatonic properties indeed.
Because of impurity equality of diatonic fifths and major thirds, the auditory tuning of the Well Tempered Meantone is quite easy, with some slight complication on the fifths on altered notes. The auditory tuning of Kirnberger III has the advantage it is easier still to tune, with characteristics very close to those of other very plausible temperaments holding a Bach attribute. This would have allowed Kirnberger to pretend he is in line with Bach's teaching, what was also testified by Forkel, and not contradicted by Bach's suns.

[4] Conclusion

Based on the ranking of table 6, and the published strong historical and musicological evidence assessed and provided by Kelletat, also Lindley, it can be assumed that:

- The proposed impurity measure paragraph [3] can be used for a comprehensive evaluation of well temperaments, by ranking of temperaments according to the obtained impurity figures. This impurity measure is easy to calculate.
  The temperaments at the top of the ranking are equivalent, in case a choice of a well temperament is necessary.
- This ranking leads to the observation of a strong fit with Kelletat's and Lindley's investigations and assumptions, specifically regarding possible candidate Bach temperaments.
- The Well Tempered Meantone of table 5 looks like a very plausible "Bach temperament" candidate:
  - It is comparable to Kirnberger III, also concerning the semi tones
  - Its auditory tuning is easy (equality of beating rates)
  - It has the best impurity qualities
  - Its fifths characteristics also fit well with scrolls on a score of J. S. Bach's WTC
  - It looks as if derived from the meantone (1/5 s.c.)

[5] Appendix 1 Well Temperament with best possible diatonic purity

Following sum must be minimized:

$$\text{Quadratic Sum} = p_C^2 + p_D^2 + p_E^2 + p_F^2 + p_G^2 + p_A^2 \quad + \quad q_C^2 + q_F^2 + q_G^2$$

Worked out in function of the notes, one obtains:

$$\begin{aligned}\text{Quadratic sum} = {} & 34F_3^2 + 38C_4^2 + 50G_3^2 + 13D_4^2 + 41A_3^2 + 29E_4^2 + 32B_3^2 \\ & -12F_3C_4 - 40F_3A_3 - 24C_4G_3 - 40C_4E_4 - 12G_3D_4 - 40G_3B_3 \\ & -24D_4A_. - 12A_3E_4 - 24E_4B_3\end{aligned}$$

The coefficients of the partial derivatives set to zero are given in the table below:

N	F3	C4	G3	D4	E4	B3	=	A3
F3	34	-6	0	0	0	0	=	20
C4	-3	19	-6	0	-10	0	=	0
G3	0	-6	25	-3	0	-10	=	0
D4	0	0	-6	13	0	0	=	12
E4	0	-20	0	0	29	-12	=	6
B3	0	0	-10	0	-6	16	=	0

The obtained B and F pitches (= as solution of the equations of the above matrix ; see further the table below) impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{246.57}{175.86} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500457\ldots$$

The above ratio is slightly above perfection, but so little that it still can be acceptable within a well temperament. Further creation of an “optimal” well temperament therefor, only allows for even distribution of this minute obligate fifths augmentation over the six remaining fifths.

$$q_{Note} = q_B = q_{F\#} = q_{C\#} = q_{G\#} = q_{Eb} = q_{Bb}$$

The collection of solutions leads to the following scale :

	F3	F#3	G3	G#3	A3	B*b*3	B3	C4	C#4	D4	E*b*4	E4
f_{Note}	175.86	184.98	197.08	208.24	220.00	234.40	246.57	263.19	277.58	294.04	312.47	329.06
q_{Note}	–1.19	0.22	–3.16	0.22	–1.89	0.22	0.22	–1.26	0.22	–2.11	0.22	–0.89
p_{Note}	0.72	12.71	0.89	11.56	10.32	4.12	17.02	0.28	18.96	9.67	14.29	20.63

[6] Appendix 2 Well Temperament with optimal auditory tuned diatonic purity

Beatings are normally negative on fifths (too small), and positive on major thirds (too large). Normally the absolute average beating rate “M” therefore is:

$$M = \frac{-q_F - q_C - q_G - q_D - q_A - q_E \quad + \quad p_F + p_C + p_G}{9}$$

Taking the signs into account, the single beating rate deviations from the mean beating rate are :

Fifths : $\Delta_{\mathrm{Fi;Note}} = -\mathrm{q}_{\mathrm{Note}} - \mathrm{M}$ Major Thirds : $\Delta_{\mathrm{Th;Note}} = \mathrm{p}_{\mathrm{Note}} - \mathrm{M}$

The appropriate sum of the squares of deviations becomes :

$$\sum \Delta^2_{\mathrm{Fi\ and\ T;Note}} = \Delta^2_{\mathrm{Fi;F}} + \Delta^2_{\mathrm{Fi;C}} + \Delta^2_{\mathrm{Fi;G}} + \Delta^2_{\mathrm{Fi;D}} + \Delta^2_{\mathrm{Fi;A}} + \Delta^2_{\mathrm{Fi;E}} + \Delta^2_{\mathrm{Th;F}} + \Delta^2_{\mathrm{Th;C}} + \Delta^2_{\mathrm{Th;G}}$$

The elaboration of this sum in function of the notes leads to :

$$\begin{aligned} 81 \times \sum \Delta^2_{Fi\ and\ T;Note} = {} & 2718F_3^2 + 2934C_4^2 + 3726G_3^2 + 1044D_4^2 + 3240A_3^2 + 2124E_4^2 + 2592B_3^2 \\ & -1116\mathrm{F_3C_4} - 216\mathrm{F_3G_3} + 36\mathrm{F_3D_4} - 3132\mathrm{F_3A_3} + 180\mathrm{F_3E_4} \\ & -2376\mathrm{C_4G_3} + 72\mathrm{C_4D_4} + 216\mathrm{C_4A_3} - 2880\mathrm{C_4E_4} \\ & -864\mathrm{G_3D_4} + 324\mathrm{G_3A_3} + 540\mathrm{G_3E_4} - 3240\mathrm{G_3B_3} \\ & -1998\mathrm{D_4A_3} - 90\mathrm{D_4E_4} - 1242\mathrm{A_3E_4} - 1944\mathrm{E_4B_3} \end{aligned}$$

The table below displays the simplified coefficients of equations obtained from the partial derivatives to the notes :

	F3	C4	G3	D4	E4	B3	=	A3
∂/∂F3 :	151	– 31	– 6	1	5	0	=	87
∂/∂C4 :	– 31	163	– 66	2	– 80	0	=	– 6
∂/∂G3 :	– 2	– 22	69	– 8	5	– 30	=	– 3
∂/∂D4 :	2	4	– 48	116	– 5	0	=	111
∂/∂E4 :	10	– 160	30	– 5	236	– 108	=	69
∂/∂B3 :	0	0	– 5	0	– 3	8	=	0

Calculation of diatonic notes for C – major

The obtained B and F pitches (= as solution of the equations of the above table ; see the main text, par. [2.3], table 4) impose that the remaining six fifths have to be slightly augmented. This can be mathematically expressed by :

$$\frac{B3}{F3} \times (fifth)^6 \times 2^n = \frac{246.22}{175.67} \times (fifth)^6 \times 2^{-4} = 1 \quad \text{and therefore} \quad fifth = 1.500545\ldots$$

The above ratio is slightly above perfection, but so little that it still can be acceptable within a well temperament. Further creation of an "optimal" well temperament therefor, only allows for even distribution of this minute obligate fifths augmentation over the six remaining fifths.

$$\mathrm{q}_{\mathrm{Note}} = \mathrm{q}_{\mathrm{B}} = \mathrm{q}_{\mathrm{F\#}} = \mathrm{q}_{\mathrm{C\#}} = \mathrm{q}_{\mathrm{G\#}} = \mathrm{q}_{\mathrm{Eb}} = \mathrm{q}_{\mathrm{Bb}}$$

The collection of solutions leads to the scale, displayed in the main text, par. [2.3], table 4.

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[A] Wohltemperierung heißt mathematisch=akustische und praktisch=musikalische Einrichtung von Tonmaterial innerhalb der zwölfstufigen Oktavskala zum einwandfreien Gebrauch in allen Tonarten auf der Grundlage des natürlich= harmonischen Systems mit dem Bestreben möglichster Reinerhaltung der diatonischen Intervalle.
Sie tritt auf als proportionsgebundene, sparsam temperierende Lockerung und Dehnung des mitteltönigen Systems, als ungleichschwebende Semitonik und als gleichschwebende Temperatur.